

Computational Game Theory

Coalitional Games and the Core

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2023

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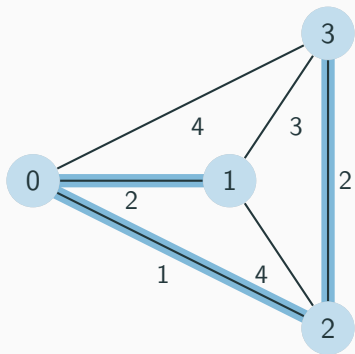
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How to share the cost?

- Cities 1, 2, 3 need to connect to the *provider* of energy 0
- The graph shows costs of pairwise connections
- For each $A \subseteq \{1, 2, 3\}$, the cost of connecting A to 0 is determined by the **minimum cost spanning tree** on $A \cup \{0\}$



$$c(A) = \begin{cases} 2 & A = 1 \\ 1 & A = 2 \\ 4 & A = 3 \\ 3 & A = 12, 23 \\ 5 & A = 13, 123 \end{cases}$$

How to determine the voting power?

UN Security Council

- 5 *permanent* and 10 *non-permanent* members
- A resolution is approved by all the permanent members and at least 4 non-permanent members
- This voting system can be represented by a function

$$v(A) = \begin{cases} 1 & \text{if } A \supseteq \{1, \dots, 5\} \text{ and } |A| \geq 9, \\ 0 & \text{otherwise,} \end{cases}$$

where $A \subseteq N = \{1, \dots, 15\}$

- The coalition A with $v(A) = 1$ is called *winning*

What is it all about?

- **Fair** division of costs
- **Fairness** of a complicated voting system
- **Power** of agents controlling some resources
- **Efficient** allocation of the profit among agents
- **Influence** of features in a predictive ML model

Game forms

1. Normal (Strategic)
2. Extensive
3. Coalitional

Games in coalitional form

- Players can create *coalitions*
- A coalition is a set of players coordinating their strategies in order to maximize *the utility of the coalition*
- Strategic aspects of coalitional games are unimportant, since they are implicitly part of the deal among players

Players and coalitions

- The **player set** is

$$N = \{1, \dots, n\}, \quad \text{for some } n \in \mathbb{N}$$

- A **coalition** is a subset $A \subseteq N$, where
 - \emptyset is the empty coalition
 - N is the *grand coalition*
 - $\{i\}$ is a one-player coalition
- The set of all coalitions is the **powerset**

$$\mathcal{P}(N) = \{A \mid A \subseteq N\}$$

Notation: we write simply 238 to denote the coalition $\{2, 3, 8\}$

Coalitional games

Definition

Coalitional game is a pair (N, v) , where v is a function

$$v: \mathcal{P}(N) \rightarrow \mathbb{R} \quad \text{such that } v(\emptyset) = 0.$$

- Coalition A receives the **worth** (incurs the **cost**) $v(A)$ independently of the actions of players in $N \setminus A$
- We will identify a coalitional game (N, v) with function v when the player set N is understood

Properties of coalitional game v

Superadditive

$$v(A) + v(B) \leq v(A \cup B) \quad \text{if } A \cap B = \emptyset, \text{ for all } A, B \subseteq N$$

Monotone

$$v(A) \leq v(B) \quad \text{if } A \subseteq B, \text{ for all } A, B \subseteq N$$

Simple

$$v(A) \in \{0, 1\} \text{ for each } A \subseteq N, \quad v(N) = 1, \quad \text{and monotone}$$

Main questions of coalitional game theory

1. **Which** coalitions will form?
2. **How** a coalition allocates its worth to its members?

Which coalitions will form?

- A **coalitional structure** is a partition $\mathcal{S} = \{A_1, \dots, A_k\}$ of N :
 1. $A_1 \cup \dots \cup A_k = N$, where $A_i \neq \emptyset$
 2. $A_i \cap A_j = \emptyset$ for all $i \neq j$
- The total utility of \mathcal{S} is then

$$V(\mathcal{S}) = \sum_{i=1}^k v(A_i)$$

Coalition formation problem

Find a coalitional structure \mathcal{S}^* satisfying

$$V(\mathcal{S}^*) = \max \{V(\mathcal{S}) \mid \mathcal{S} \text{ is a coalitional structure}\}$$

Coalition formation problem

- **Bell numbers** B_n count the number of coalitional structures:

n	3	...	10	...	15
B_n	15	...	115 975	...	1 382 958 545

- Finding an optimal coalitional structure \mathcal{S}^* is NP-complete

Trivial solution for superadditive games

Let v be a superadditive game. For any coalitional structure \mathcal{S} ,

$$V(\mathcal{S}) = \sum_{i=1}^k v(A_i) \leq v(N) = V(\{N\}).$$

This implies that $\mathcal{S}^* = \{N\}$.

Main questions revisited

Which coalitions will form?

- We assume that players form grand coalition N
- This is optimal for superadditive games

How a coalition allocates its worth to its members?

- An **allocation** is a vector $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$
- If \mathbf{x} is allocated to players, coalition $A \subseteq N$ obtains

$$\mathbf{x}(A) = \sum_{i \in A} x_i$$

The solution for a game v is a set of allocations $\mathbf{x} \in \mathbb{R}^n$.

Core

What is the core of a game?

The core is a set of efficient allocations upon which no coalition can improve.

Definition

The **core** of a game v is the set

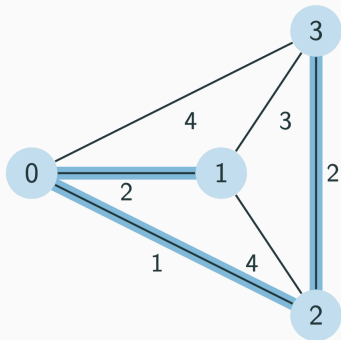
$$\mathcal{C}(v) = \{ \mathbf{x} \in \mathbb{R}^n \mid \underbrace{\mathbf{x}(N) = v(N)}_{\text{Efficiency}}, \underbrace{\mathbf{x}(A) \geq v(A), \forall A \subseteq N}_{\text{Coalitional rationality}} \}.$$

- The core is a convex polytope in \mathbb{R}^n of dimension $\leq n - 1$
- Is the core always nonempty? How to find core allocations?

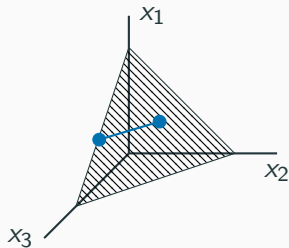
How to divide the cost – a solution

The core has reversed inequalities in case of cost games:

$x_1 \leq 2$, $x_2 \leq 1$, $x_3 \leq 4$, $x_1 + x_2 \leq 3$, $x_1 + x_3 \leq 5$, $x_2 + x_3 \leq 3$, and
 $x_1 + x_2 + x_3 = 5$



$$C(c) = \text{conv}\{(2, 0, 3), (2, 1, 2)\}$$



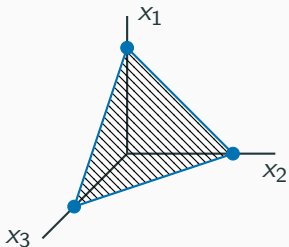
Games can have empty cores

Simple majority voting

Three players vote by majority. This determines a game

$$v(A) = \begin{cases} 1 & |A| \geq 2, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for all } A \subseteq \{1, 2, 3\}.$$

Then $\mathcal{C}(v) = \emptyset$.



How to decide nonemptiness of the core

$$\mathcal{C}(v) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}(N) = v(N), \quad \mathbf{x}(A) \geq v(A), \quad \forall A \subseteq N\}$$

Linear program with real variables x_1, \dots, x_n

Minimize $x_1 + \dots + x_n$

subject to $\sum_{i \in A} x_i \geq v(A)$ for each nonempty $A \subseteq N$

The following are equivalent:

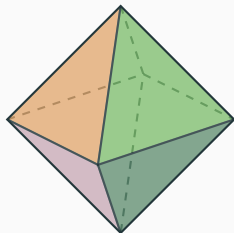
- The optimal value is $v(N)$
- $\mathcal{C}(v) \neq \emptyset$

How to find all core allocations

The core $\mathcal{C}(v)$ has a representation

$$\mathcal{C}(v) = \text{conv}\{\mathbf{x}_1, \dots, \mathbf{x}_k\},$$

where $\mathbf{x}_1, \dots, \mathbf{x}_k$ are the **vertices** of $\mathcal{C}(v)$.



Vertex enumeration problem

- Find *all* vertices of the core $\mathcal{C}(v)$
- A hard problem studied in polyhedral geometry

This problem has a closed-form solution for some games.

Games with incentives to join large coalitions

A game v is **supermodular** if

$$v(A) + v(B) \leq v(A \cup B) + v(A \cap B) \quad \text{for all } A, B \subseteq N.$$

Proposition

The following are equivalent.

- Game v is supermodular.
- For all $A, B \subseteq N$ with $A \subseteq B$, and each $i \in N \setminus B$,

$$v(A \cup i) - v(A) \leq v(B \cup i) - v(B).$$

It is about marginal contributions of players

- Given a **permutation** π of N , the rank of player i is $\pi(i)$
- The coalition preceding player i is then

$$A_i^\pi = \{j \in N \mid \pi(j) < \pi(i)\}.$$

Definition

A **marginal vector** is an allocation $\mathbf{x}^\pi \in \mathbb{R}^n$ such that

$$x_i^\pi = v(A_i^\pi \cup i) - v(A_i^\pi), \quad i \in N.$$

Example: Marginal vectors in a supermodular game

This three-player game is supermodular: $v(A) = \begin{cases} 0 & |A| \leq 1, \\ 1 & |A| = 2, \\ 3 & |A| = 3. \end{cases}$

Permutation	Marginal vector
123	(0, 1, 2)
132	(0, 2, 1)
213	(1, 0, 2)
231	(2, 0, 1)
312	(1, 2, 0)
321	(2, 1, 0)

Observe that each marginal vector is a core allocation.

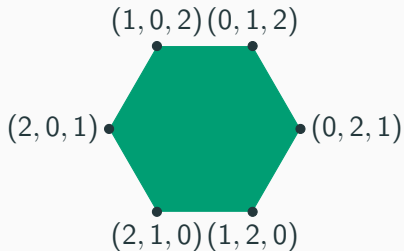
Cores of supermodular games

Theorem

The following are equivalent.

- Game v is supermodular
- The vertices of $\mathcal{C}(v)$ are precisely marginal vectors

$$v(A) = \begin{cases} 0 & |A| = 1, \\ 1 & |A| = 2, \\ 3 & |A| = 3. \end{cases}$$



Summing up the core properties

Pros

- Simple definition
- Core allocations are stable
- Known for some games

Cons

- May be empty
- May be large
- Hard to compute

We can seek solution concepts based on different criteria:

- Nonemptiness
- Single allocation
- **Fairness**