# Computed tomography (CT)

## Part 2

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2005–2022

[1] Using images from J. Hozman, J. Fessler, S. Webb, M. Slaney, A. Kak and others

Analytical methods

Algebraic reconstruction

3D CT

Radiation dose

# Reconstruction methods

- *Backprojection* (not an inverse)
- *Fourier reconstruction* (slow)
- Filtered backprojection
- Algebraic reconstruction (iterative)

# Forward projection

## sinogram

$$P_\varphi(r) = \int_{(x,y)\in L(r,\varphi)} \mu(x,y)\mathrm{d}l$$

$$r = x\cos\varphi + y\sin\varphi$$

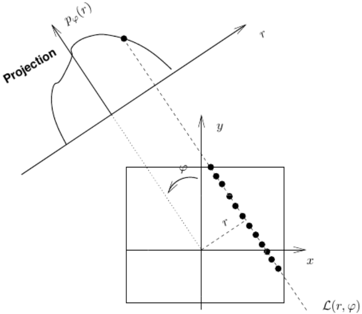
$$P_\varphi(r) = \int_t o(x,y)\mathrm{d}t$$

$$x = r\cos\varphi - t\sin\varphi$$

$$y = r\sin\varphi + t\cos\varphi$$

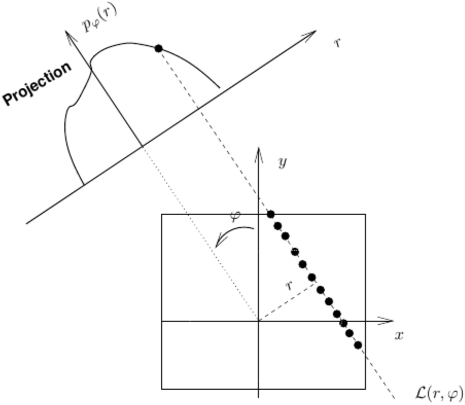
Variable correspondence:

$$\xi' = r, \quad \eta' = t, \quad \xi = x, \quad \eta = y$$



# Backprojection

## laminogram




$$\mu_b(x,y) = \int_0^\pi P_\varphi(r)\mathrm{d}\varphi$$

$$r = x\cos\varphi + y\sin\varphi$$

# Backprojection

## laminogram

$$\mu_b(x, y) = \int_0^{\pi} P_\varphi(r)\mathrm{d}\varphi$$

$$r = x\cos\varphi + y\sin\varphi$$

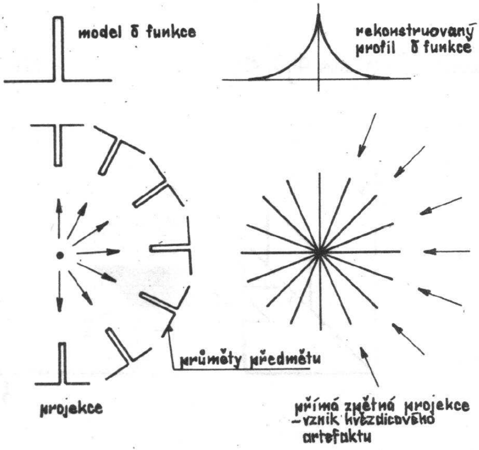
for uniformly discretized $\varphi$

$$\varphi_i = \pi(i-1)/n_\varphi, \quad i = 1, \ldots, n_\varphi$$

$$\mu_b(x, y) \approx \frac{\pi}{n_\varphi}\sum_{i=1}^{n_\varphi} P_\varphi(x\cos\varphi_i + y\sin\varphi_i)$$

# Backprojection

...is **not** an inverse of the Radon transform, leads to *star artifacts*




*laminogram* $\mu_b$ — the original object $\mu$ blurred, convolved by $1/|r|$

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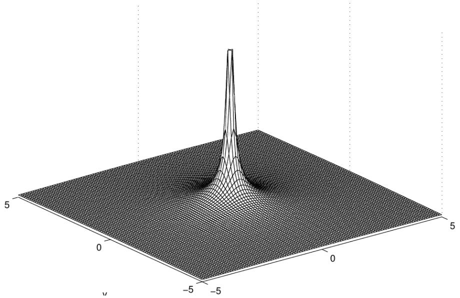


*laminogram* $\mu_b$ — the original object $\mu$ blurred, convolved by $1/|r|$

# Central slice theorem

(Projection Theorem, Věta o centrálním řezu)

$$P_\varphi(r) = \int \mu(r\cos\varphi - t\sin\varphi, r\sin\varphi + t\cos\varphi)\mathrm{d}t$$

Fourier transform of the Radon transform by $r$:

$$\begin{aligned}\mathscr{F}\left\{\mathscr{R}\left[\mu(x,y)\right]\right\} = \mathscr{F}\left\{P_\varphi(r)\right\} = \hat{P}_\varphi(\omega) &= \int P_\varphi(r)\mathrm{e}^{-2\pi j\omega r}\mathrm{d}r \\ &= \iint \mu(r\cos\varphi - t\sin\varphi, r\sin\varphi + t\cos\varphi)\mathrm{e}^{-2\pi j\omega r}\mathrm{d}r\mathrm{d}t\end{aligned}$$

Substitution $(r,t) \to (x,y)$:

$$\hat{P}_\varphi(\omega) = \int \mu(x,y)\mathrm{e}^{-2\pi j\omega(x\cos\varphi + y\sin\varphi)}\mathrm{d}x\mathrm{d}y$$

# Central slice theorem

$$\hat{P}_\varphi(\omega) = \int \mu(x,y)\mathrm{e}^{-2\pi j\omega(x\cos\varphi + y\sin\varphi)}\mathrm{d}x\mathrm{d}y$$

Denote $u = \omega\cos\varphi \quad v = \omega\sin\varphi$

$$\hat{P}(u,v) = \int \mu(x,y)\mathrm{e}^{-2\pi j(xu+yv)}\mathrm{d}x\mathrm{d}y$$

and therefore

$$\begin{aligned} \hat{P}(u,v) &= \mathscr{F}\left\{\mu(x,y)\right\} \\ \hat{P}_\varphi(\omega) &= \mathscr{F}\left\{\mu(x,y)\right\}(\omega\cos\varphi, \omega\sin\varphi) = \hat{\mu}(\omega\cos\varphi, \omega\sin\varphi) \end{aligned}$$

# Central slice theorem

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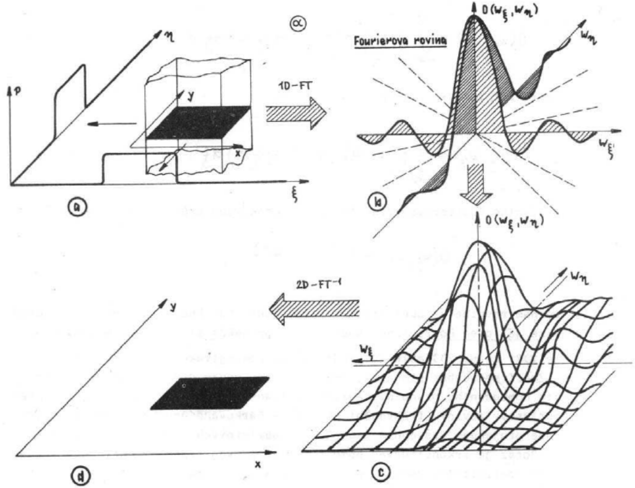
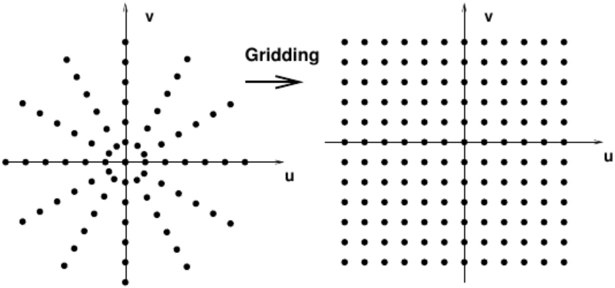
$$\hat{P}_\varphi(\omega) = \mathscr{F}\left\{\mu(x, y)\right\}(\omega\cos\varphi, \omega\sin\varphi) = \hat{\mu}(\omega\cos\varphi, \omega\sin\varphi)$$

Slice of the 2D Fourier transform of the image $\mu$ at angle $\varphi$ is the 1D Fourier transform of the projection $P_\varphi$ of the same image $\mu$.

# Fourier reconstruction



# Fourier reconstruction (2)




- 1D FT $\hat{P}_\varphi(\omega)$ of each projection $P_\varphi(r)$
- Interpolate FT from polar to Cartesian grid (to get $\hat{P}(u, v)$)
- Inverse 2D FT $\hat{P}(u, v)$ to get object $\mu$

Cons: computational complexity, interpolation artifacts

# Inverse Radon transform

From the Fourier slice theorem:

$$\hat{P}(u,v) = \mathscr{F}\left\{\mu(x,y)\right\}$$

$$\mu(x,y) = \mathscr{F}^{-1}\left\{\hat{P}(u,v)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{P}(u,v)\mathrm{e}^{2\pi j(xu+yv)}\mathrm{d}u\mathrm{d}v$$

Polar coordinates $u = \omega\cos\varphi, \quad v = \omega\sin\varphi$:

$$\mu(x,y) = \int_{0}^{\pi}\int_{-\infty}^{\infty} \hat{P}_{\varphi}(\omega)\mathrm{e}^{2\pi j\omega(x\cos\varphi+y\sin\varphi)}|\omega|\mathrm{d}\omega\mathrm{d}\varphi$$

where $|\omega|$ is the Jacobian (determinant) of $(\omega,\phi)\to(u,v)$

$$\begin{vmatrix} \frac{\partial u}{\partial\varphi} & \frac{\partial u}{\partial\omega} \\ \frac{\partial v}{\partial\varphi} & \frac{\partial v}{\partial\omega} \end{vmatrix} = \left|-\omega\sin^2\varphi - \omega\cos^2\varphi\right| = \left|\omega\right|$$

# Inverse Radon transform

$$\mu(x, y) = \int_0^\pi \int_{-\infty}^\infty \hat{P}_\varphi(\omega) \mathrm{e}^{2\pi j\omega(x\cos\varphi + y\sin\varphi)} |\omega| \mathrm{d}\omega \mathrm{d}\varphi$$

can be written as

$$\mu(x, y) = \int_0^\pi Q_\varphi(\underbrace{x\cos\varphi + y\sin\varphi}_{r}) \mathrm{d}\varphi$$

$$Q_\varphi(r) = \int_{-\infty}^\infty \hat{P}_\varphi(\omega) \mathrm{e}^{2\pi j\omega r} |\omega| \mathrm{d}\omega$$

where $Q_\varphi(r)$ is a modified projection

# Inverse Radon transform

$$\mu(x, y) = \int_0^\pi Q_\varphi(r)\mathrm{d}\varphi$$

$$Q_\varphi(r) = \int_{-\infty}^{\infty} \hat{P}_\varphi(\omega)\mathrm{e}^{2\pi j\omega r}|\omega|\mathrm{d}\omega$$

$$Q_\varphi(r) = \mathscr{F}^{-1}\left\{|\omega|\hat{P}_\varphi(\omega)\right\} = \mathscr{F}^{-1}\left\{|\omega|\right\} * P_\varphi(r)$$

defining the exact inverse Radon transform

$$P_\varphi(r) = \mathscr{R}\left[\mu(x, y)\right]$$
$$\mu(x, y) = \mathscr{R}^{-1}\left[P_\varphi(r)\right]$$

# Filtered backprojection

## Filtrovaná zpětná projekce

- Filter all projections $P_\varphi(r)$ for all $\varphi$, get modified projections $Q_\varphi(r)$
- Backproject modified projections and sum

$$\mu(x,y) = \int_0^\pi Q_\varphi(r)\mathrm{d}\varphi$$

$$Q_\varphi(r) = h(t) * P_\varphi(r) = \mathscr{F}^{-1}\left\{H(\omega)\right\} * P_\varphi(r)$$

$$H(\omega) = |\omega|$$

- No Fourier transform involved.

# Practical implementation of filtered backprojection

- **Problem:** Ideal filter $H(\omega) = |\omega|$ amplifies noise
- **Solution:** Make $\hat{P}_\varphi(\omega)$ frequency limited.
  Ramakrishnan-Lakshiminaryanan $\longrightarrow$ Ram-Lak filter:

$$H(\omega) = \begin{cases} |\omega| & \text{if } |\omega| \leq \Omega \\ 0 & \textit{otherwise} \end{cases}$$

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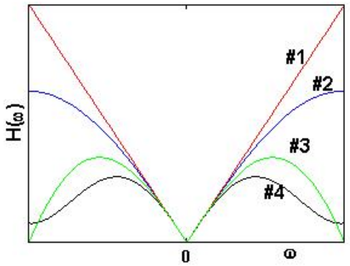
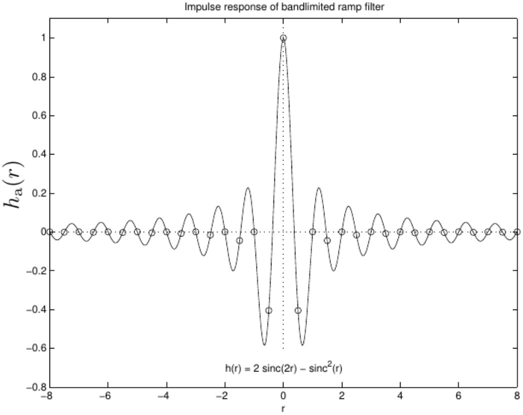
- Ram-Lak filter causes artefacts (Gibbs). Many solutions (Hamming filter, Shepp-Logan filter). Tradeoff between SNR and resolution.

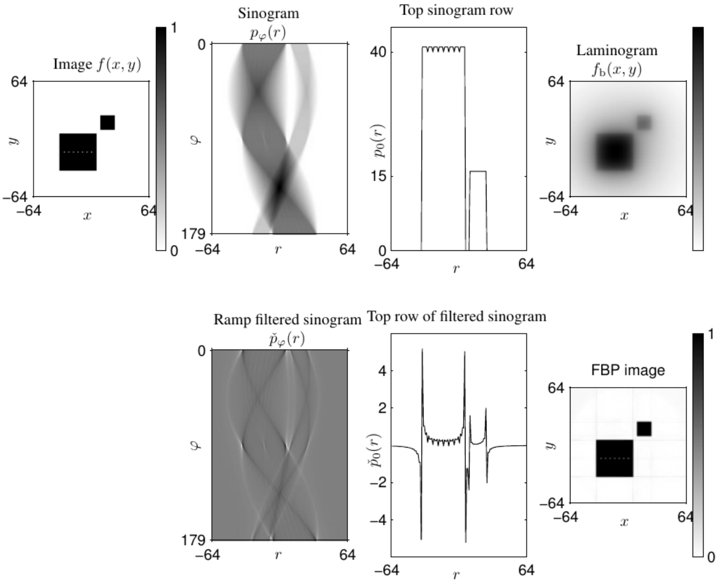


# Bandlimited ramp filter $h$

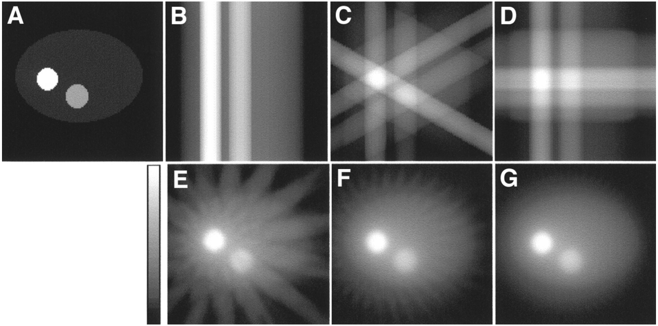
in space domain



# Filtered backprojection example



# Filtered backprojection




original image, 1,3, 4, 16, 32, a 64 projections

# Fan-beam reconstruction

- Rays not parallel, not a Radon transform.
- Rebinning

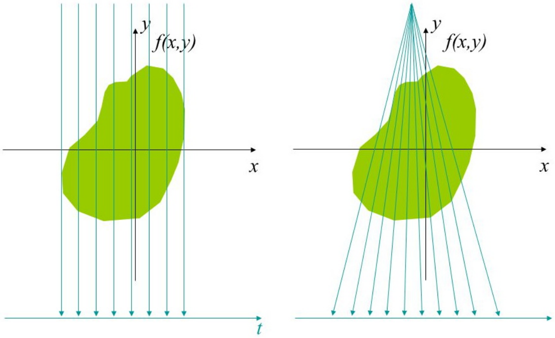



image courtesy of Gillian Henderson

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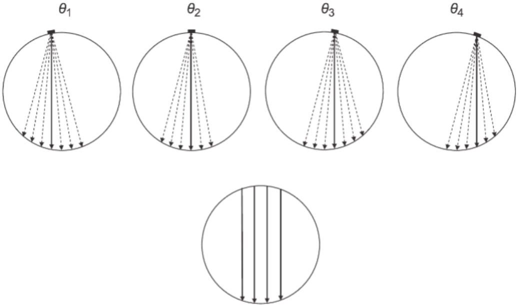


image courtesy of Jonathan Mamou and Yao Wang

# Fan-beam reconstruction (2)

- Rays not parallel, not a Radon transform.
- Exact algorithms:
  - Rebinning
  - filtered backprojection (Katsevich) — computational complexity, increased dose.
- Approximate algorithms: Modified filtered backprojection (quadratic cosine correction, $\cos\theta$). Feldkamp-Davis-Kress

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- Algebraic reconstruction. Best quality but slow.

# Algebraic reconstruction

- Setup and solve a (large) system of equations describing the measurements.
- Mostly (but not necessarily) linear

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**Advantages over FBP**

- Better modeling of the physics — attenuation, scattering, limited resolution, beam geometry, sensor noise, beam hardening. . .
- Flexible, better handling of limited acquisition — restricted region, restricted angles, few measurements required
- Can use a statistical image model (regularization)
- Higher quality, less apparent artifacts

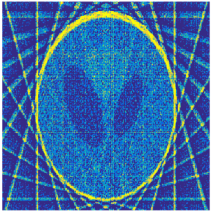
**Disadvantage** — speed

# FBP versus ART

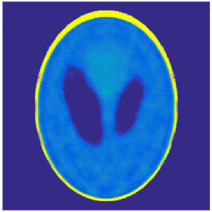
## few projections

**Phantom**

**FBP** `(iradon)`

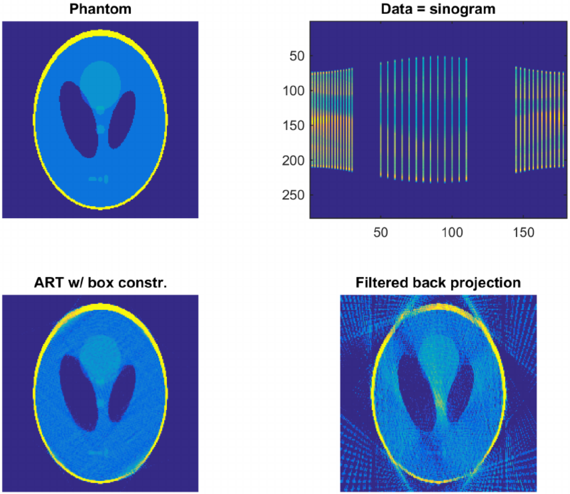


**ART w/ box constraints**



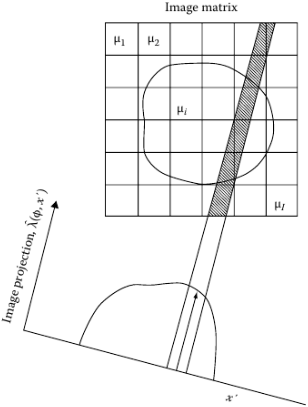
Courtesy of Technical University of Denmark

# FBP versus ART
## missing angles




Courtesy of Technical University of Denmark

# Linear reconstruction




- Discretize continuous $\mu(\mathbf{x})$ to pixels $\mu_i$

$$\mu(\mathbf{x}) = \sum_{i=1}^{M} \mu_i \psi_i(\mathbf{x})$$

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- Basis functions (piecewise constant, P0)

$$\psi_i(\mathbf{x}) = \begin{cases} 1, \text{if } \mathbf{x} \text{ in pixel } i \\ 0, \text{otherwise} \end{cases}$$

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- Radon transform

$$P_\varphi(r) = \mathscr{R}[\mu](\varphi, r) = \sum_{i=1}^{M} \mu_i \mathscr{R}[\psi_i](\varphi, r)$$

# Linear reconstruction (2)

- For all projections $p_j = P_{\varphi_j}(r_j)$, $j = 1, \ldots, N$

$$p_j = P_{\varphi_j}(r_j) = \sum_{i=1}^{M} \mu_i \underbrace{\mathscr{R}[\psi_i](\varphi_j, r_j)}_{w_{ij}}$$

$$p_j = \sum_{i=1}^{M} w_{ij} \mu_i$$

$$\mathbf{p} = \mathrm{W} \boldsymbol{\mu}$$

where $\mu_i$ are pixel values, $p_j$ are the projections.

Knowing $\mathbf{p}$, solve for $\boldsymbol{\mu}$.

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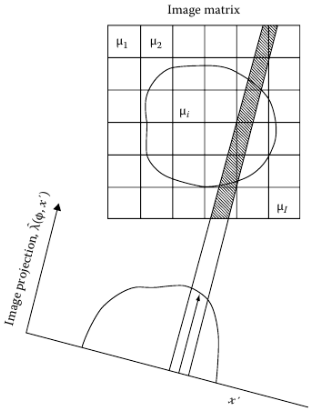
  Knowing $\mathbf{p}$, solve for $\boldsymbol{\mu}$.

- Linear equation system
  - is big ($10^4 \sim 10^6$ unknowns and measurements)
  - can be overdetermined
  - can be underdetermined
  - is sparse

# Weight coefficients




For line rays — intersection length

$$w_{ij} = \int_{\mathbf{x} \in L(r_j, \varphi_j)} \psi_i(\mathbf{x}) \mathrm{d}l$$

# Weight coefficients

For line rays — intersection length

$$w_{ij} = \int_{\mathbf{x} \in L(r_j, \varphi_j)} \psi_i(\mathbf{x}) \mathrm{d}l$$

For thick rays — intersection area

$$w_{ij} = \int_{\mathbf{x} \in L'(r_j, \varphi_j)} \psi_i(\mathbf{x}) \mathrm{d}\mathbf{x}$$

# Weight coefficients

For line rays — intersection length

$$w_{ij} = \int\limits_{\mathbf{x} \in L(r_j, \varphi_j)} \psi_i(\mathbf{x}) \mathrm{d}l$$

Binary approximation

$$w_{ij} = \begin{cases} 1, & \text{if ray } L(r_j, \varphi_j) \text{ intersects pixel } \psi_i \\ 0, & \text{otherwise} \end{cases}$$

# Least squares solution

## for overdetermined systems

Minimize the reconstruction error $\mathbf{e}$

$$\boldsymbol{\mu}^* = \arg\min_{\boldsymbol{\mu}} \|\underbrace{\mathsf{W}\boldsymbol{\mu} - \mathbf{p}}_{\mathbf{e}}\|^2$$

# Least squares solution

## for overdetermined systems

Minimize the reconstruction error **e**

$$\boldsymbol{\mu}^* = \arg\min_{\boldsymbol{\mu}} \|\underbrace{\mathsf{W}\boldsymbol{\mu} - \mathbf{p}}_{\mathbf{e}}\|^2$$

The reconstruction error **e** must be perpendicular to *range* of W.

$$0 = \mathsf{W}^T\mathbf{e} = \mathsf{W}^T(\mathsf{W}\boldsymbol{\mu}^* - \mathbf{p})$$

Normal equations

$$\mathsf{W}^T\mathbf{p} = \mathsf{W}^T\mathsf{W}\boldsymbol{\mu}^*$$

Pseudoinverse solution

$$\boldsymbol{\mu}^* = (\mathsf{W}^T\mathsf{W})^{-1}\mathsf{W}^T\mathbf{p}$$

suitable for smaller problems

# Minimum-norm solution

## for underdetermined systems or noisy data

Add regularization D

$$\boldsymbol{\mu}^* = \arg\min_{\boldsymbol{\mu}} \|\underbrace{\mathsf{W}\boldsymbol{\mu} - \mathbf{p}}_{\mathbf{e}}\|^2 + \lambda\|\mathsf{D}\boldsymbol{\mu}\|^2$$

Normal equations

$$\mathsf{W}^T\mathbf{p} = \left(\mathsf{W}^T\mathsf{W} + \lambda\mathsf{D}^T\mathsf{D}\right)\boldsymbol{\mu}^*$$

Pseudoinverse solution

$$\boldsymbol{\mu}^* = \left(\mathsf{W}^T\mathsf{W} + \lambda\mathsf{D}^T\mathsf{D}\right)^{-1}\mathsf{W}^T\mathbf{p}$$

# Iterative methods

**Principles**

- Start from an initial guess of $\boldsymbol{\mu}$
- Compare measured projections and simulations
- Correct pixel values to decrease the difference
- Iterate until convergence

**Properties**

- Take advantage of the sparseness (complexity $O(N)$ per iteration)
- Low memory complexity ($O(M)$)
- $\longrightarrow$ Suitable for large systems of equations
- Early stopping
- Slower for small problems (compared to direct methods)

# Projection method

## Kaczmarz's method

$$p_j = \sum_{i=1}^{M} w_{ij}\mu_i, \qquad j = 1, 2, \ldots, N$$

$$p_j = \langle \mathbf{w}_j, \boldsymbol{\mu} \rangle = \mathbf{w}_j^T \boldsymbol{\mu}$$

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$$p_j = \langle \mathbf{w}_j, \boldsymbol{\mu} \rangle = \mathbf{w}_j^T \boldsymbol{\mu}$$

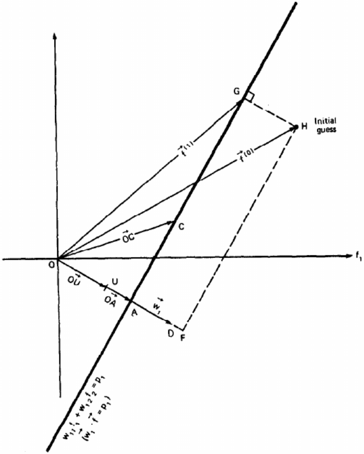
- Affine solution space of equation $j$

$$\mathcal{S}_j = \{\boldsymbol{\mu} \in \mathbb{R}^M; p_j = \langle \mathbf{w}_j, \boldsymbol{\mu} \rangle\}$$

Normal vector $\mathbf{w}_j$

$$\forall \boldsymbol{\mu} \in \mathcal{S}_j, \boldsymbol{\mu}' \in \mathcal{S}_j;\ \langle \mathbf{w}_j, \boldsymbol{\mu} - \boldsymbol{\mu}' \rangle = 0$$

# Projection to an affine space




Projection onto $\mathcal{S}_j$

$$\mathbf{g}^* = \mathcal{P}_{\mathcal{S}_j}(\mathbf{h}) = \arg\min_{\mathbf{g}\in\mathcal{S}_j} \|\mathbf{g} - \mathbf{h}\|$$

# Projection to an affine space

Projection onto $\mathcal{S}_j$

$$\mathbf{g}^* = \mathcal{P}_{\mathcal{S}_j}(\mathbf{h}) = \arg\min_{\mathbf{g}\in\mathcal{S}_j} \|\mathbf{g} - \mathbf{h}\|$$

Moving in the normal direction (minimum change) until hitting $\mathcal{S}_j$

$$\mathbf{g}^* = \mathbf{h} - \lambda \mathbf{w}_j$$
$$p_j = \langle \mathbf{w}_j, \mathbf{h} \rangle$$

Solution

$$\lambda = \frac{\langle \mathbf{w}_j, \mathbf{h} \rangle - p_j}{\langle \mathbf{w}_j, \mathbf{w}_j \rangle} \qquad \text{normalized residual}$$
$$\mathbf{g}^* = \mathbf{h} - \frac{\langle \mathbf{w}_j, \mathbf{h} \rangle - p_j}{\langle \mathbf{w}_j, \mathbf{w}_j \rangle} \mathbf{w}_j$$

# Projection method

## the algorithm

- Initial solution $\boldsymbol{\mu}^{(0)}$ (e.g. random)
- Project sequentially to constraints $1, 2, \ldots, N, 1, 2, \ldots$

$$\begin{aligned}
\boldsymbol{\mu}^{(1)} &= \mathcal{P}_{\mathcal{S}_1}\boldsymbol{\mu}^{(0)} \\
\boldsymbol{\mu}^{(2)} &= \mathcal{P}_{\mathcal{S}_2}\boldsymbol{\mu}^{(1)} \\
\boldsymbol{\mu}^{(3)} &= \mathcal{P}_{\mathcal{S}_3}\boldsymbol{\mu}^{(3)} \\
&\ldots
\end{aligned}$$

- Repeat until convergence

# Interpretation of the update

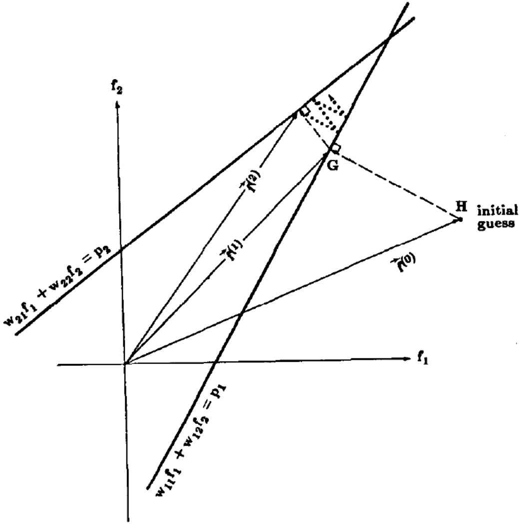
$$\boldsymbol{\mu}^{(k+1)} = \boldsymbol{\mu}^{(k)} - \underbrace{\frac{\langle \mathbf{w}_j, \boldsymbol{\mu}^{(k)} \rangle - p_j}{\langle \mathbf{w}_j, \mathbf{w}_j \rangle}}_{\tilde{p}_j} \mathbf{w}_j$$

$$p_j = \sum_{i=1}^{M} w_{ij} \mu_i = \langle \mathbf{w}_j, \boldsymbol{\mu} \rangle$$

Projection $\hat{p}_j \langle \mathbf{w}_j, \boldsymbol{\mu}^{(k)} \rangle$ along ray $j$
Backprojection of the correction $\tilde{p}_j$ along ray $j$

# Projection example

$N = 2$



# Projection method

## properties

- Computationally cheap: one projection cost $O(M)$, applying all constraints $O(MN)$
- Low-memory complexity: $O(M)$ if $\mathbf{w}_{ij}$ can be calculated on the fly.
- If a solution exists, the projection method converges to it.
- Convergence may be slow.
- If no solution exists, the method may oscillate.

# Projection method improvements

- Constraint ordering

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- Under/overrelaxation,

$$\boldsymbol{\mu} = \boldsymbol{\mu}^{(0)} - \alpha \frac{\langle \mathbf{w}_j, \boldsymbol{\mu} \rangle - p_j}{\langle \mathbf{w}_j, \mathbf{w}_j \rangle} \mathbf{w}_j$$

$$0 < \alpha < 2$$

# Projection method improvements

- Constraint ordering
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$$\boldsymbol{\mu} = \boldsymbol{\mu}^{(0)} - \alpha \frac{\langle \mathbf{w}_j, \boldsymbol{\mu} \rangle - p_j}{\langle \mathbf{w}_j, \mathbf{w}_j \rangle} \mathbf{w}_j$$

$$0 < \alpha < 2$$

- Incorporating constraints — positivity ($\mu_i \geq 0$), zero outside,…

# Simplified update rules

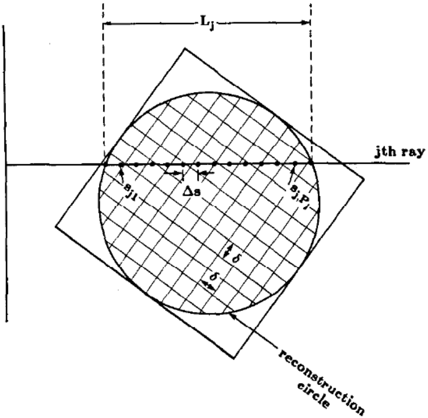
- Binary additive case ($w_{ij} \in \{0,1\}$)

$$\forall j;\ g_k^* = h_k - \frac{\sum\limits_{i,w_{ij}=1} h_i - p_j}{N_j}, \qquad \text{for } w_{kj} = 1,\ N_j = \sum_i w_{ij} = 1$$

- Binary multiplicative case ($w_{ij} \in \{0,1\}$)

$$\forall j;\ g_k^* = h_k \frac{p_k}{\sum\limits_{i,w_{ij}=1} h_i}, \qquad \text{for } w_{kj} = 1$$

# Projections by integration




$$p_j = \int \mu(r_j \cos\varphi_j - t\sin\varphi, r_j \sin\varphi_j + t\cos\varphi)\mathrm{d}t$$

$$p_j = \sum_{i=1}^{M} w_{ij}\mu_i = \langle \mathbf{w}_j, \boldsymbol{\mu} \rangle$$

# Projections by integration

$$p_j = \int \mu(r_j \cos\varphi_j - t\sin\varphi, r_j \sin\varphi_j + t\cos\varphi)\mathrm{d}t$$

$$p_j = \sum_{i=1}^{M} w_{ij}\mu_i = \langle \mathbf{w}_j, \boldsymbol{\mu} \rangle$$

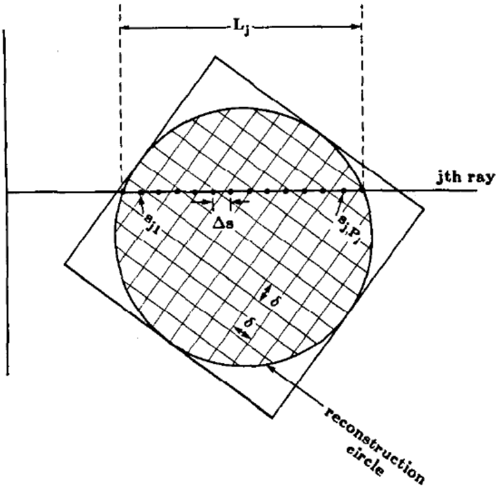
$$\mu(\mathbf{x}) = \sum_{i=1}^{M} \mu_i \psi_i(\mathbf{x})$$

$$w_{ij} = \int \psi_i(r_j \cos\varphi_j - t\sin\varphi, r_j \sin\varphi_j + t\cos\varphi)\mathrm{d}t$$

$$p_j = \Delta s \sum_k \mu(r_j \cos\varphi_j - t\sin\varphi, r_j \sin\varphi_j + t\cos\varphi),$$

with $t = \Delta s\, k$

# Backprojections by integration



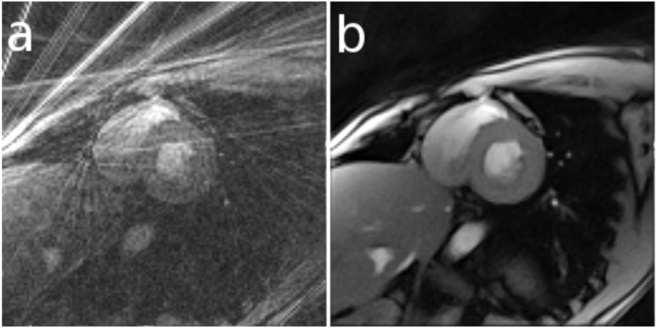

Backprojection can be also interpreted by sampling the integration path.

# Other iterative methods

- simultaneous iterative reconstruction (SIRT), Cimmino's method — block update
- simultaneous algebraic reconstruction technique (SART) — bilinear $\psi$, projection by integration, Hamming window over rays
- iterative least-squares technique (ILST)
- multiplicative algebraic reconstruction technique (MART)
- iterative sparse asymptotic minimum variance (SAMV)
- (preconditioned) conjugated gradients (CG) — needs regularization for ill-posed problems
- ...

# Example

## moving heart




filtered back projection iterative (nonlinear)

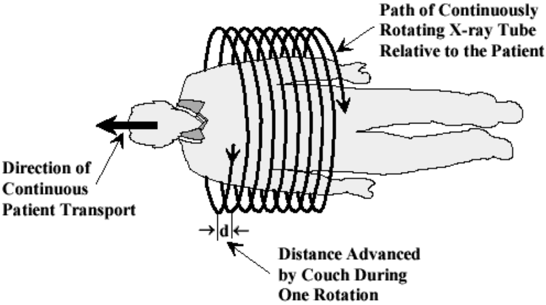
Courtesy of Biomedizinische NMR Forschungs GmbH

# 3D computed tomography

- Technical challenges: power, cooling
- Rotation method (slice by slice)
- Spiral/helix method

# Spiral method

- Acceleration: 10 min $\rightarrow$ 1 min

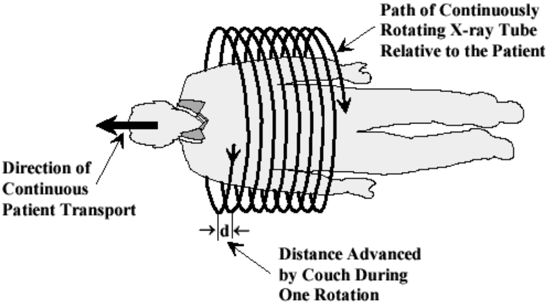


# Spiral method

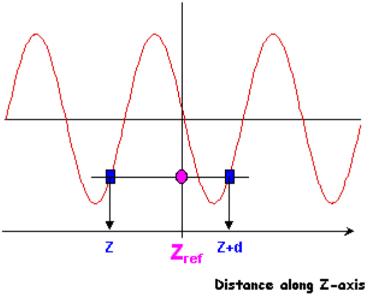
- Acceleration: 10 min $\rightarrow$ 1 min
- *Pitch*:

$$P = \Delta l / d$$

$\Delta l$ bed shift per rotation, $d$ slice thickness.
Normally $0 < P < 2$. Overlap for $P < 1$. Typically $P = 1.5$.

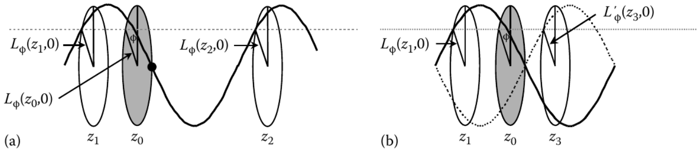


# Spiral method (2)



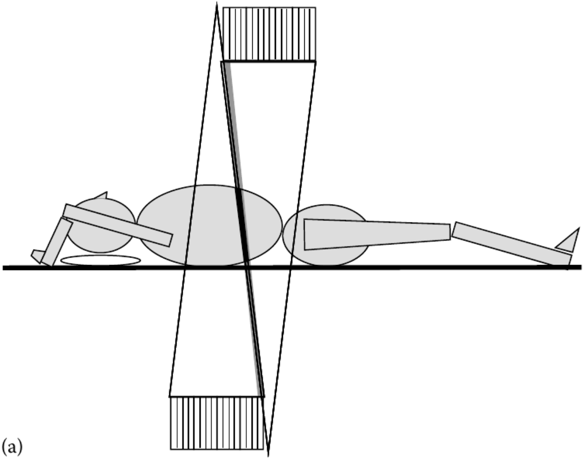

- Interpolation in $z$ axis
- Interpolation *wide* — 1 turn. Less noise, larger effective slice thickness.
- Interpolation *Slim* — 1/2 turn, symmetry. More noise, smaller effective slice thickness.

# Spiral method (2)



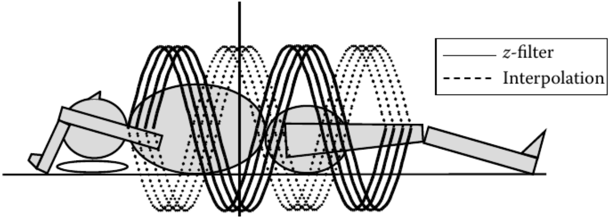

- Interpolation in $z$ axis
- Interpolation *wide* — 1 turn. Less noise, larger effective slice thickness.
- Interpolation *Slim* — 1/2 turn, symmetry. More noise, smaller effective slice thickness.

# Multislice acquisition


(a)

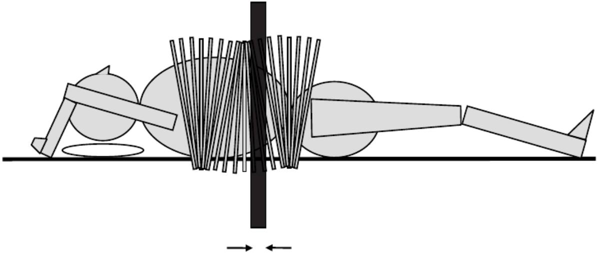
- Acceleration

# Multislice acquisition




- Multi-plane reconstruction / multi-slice linear interpolation / multi-slice filtered interpolation

# Multislice acquisition




(b)

- Multi-plane reconstruction / multi-slice linear interpolation / multi-slice filtered interpolation

# CT image quality

- Parameters:
  - Resolution (0.5 mm)
  - Contrast ($\delta H$, about $5 - 10$ HU.)
  - Detection threshold (about 1 mm at $\Delta H = 200$, 5 mm at $\Delta H = 5$).
  - Noise (SNR)
- Artifacts
  - Scanner defects, malfunctions, operator error
  - Metal parts (shadows)
  - Motion artifacts
  - Partial volume

# Artifact examples

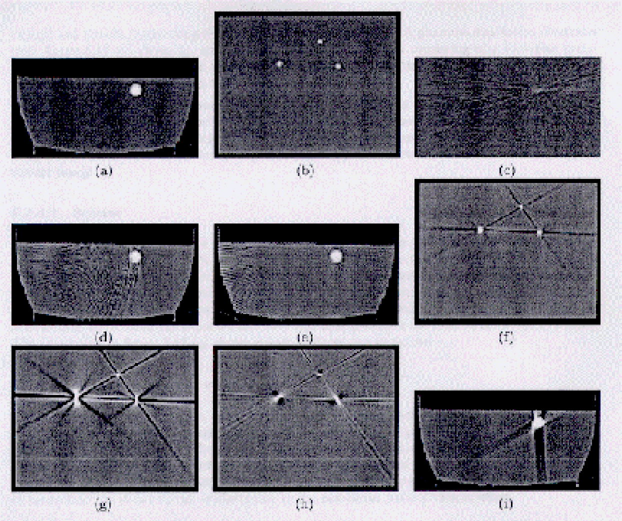



Figure 2.19 Example of image artifacts: (a) test phantom, (b) second phantom, (c) noise, (d) detector under-sampling, (e) view under-sampling, (f) beam hardening, (g) scatter, (h) nonlinear partial volume effect, and (i) object motion. (unpublished results)

Analytical methods

Algebraic reconstruction

3D CT

Radiation dose

# Radiation dose

- Absorbed dose $D$. $1\,\mathrm{Gy}$ (gray) $= 1\,\mathrm{J/kg}$ Before $1\,\mathrm{Gy} = 100\,\mathrm{rad}$
- Effective dose equivalent (dávkový ekvivalent) $H_\mathrm{E}$ [Sv] (sievert)

$$H_\mathrm{E} = \sum_i w_i H_i = \sum_i w_i c_i D_i$$

$H = cD$. Quality factor $c$ is 1 for X-rays and $\gamma$ rays, 10 for neutrons, 20 for $\alpha$ particles.

Coefficient $w$ is organ dependent: male/female glands 0.2, lungs 0.12, breast 0.1, stomach 0.12, thyroid gland 0.05, skin 0.01. $\sum w_i = 1$
Before $1\,\mathrm{Sv} = 100\,\mathrm{rem}$

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  Before $1\,\mathrm{Sv} = 100\,\mathrm{rem}$
- Sum the doses

# Radiation dose

- Medical limit (USA) is 50 mSv/year (=limit for a person working with radiation in CR), corresponding to 1000 chest X-rays, or 15 head CTs, or 5 whole body CTs (1 CT$\approx$ 10 mSv).
- low-dose CT$\approx 2 \sim 5$ mSv, PET$\approx$ 25 mSv
- In CR radioactive background about 3 mSv/year (mainly radon), similar to USA. In Colorado (altitude $1500 \sim 4000$ m) about 4.5 mSv/year. Mean dose from medical imaging 0.3 mSv/year, about 3 long flights.

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- cancer related death 20 %. 1 CT=10 mSv — relative increase by $10^{-3} \sim 10^{-4}$

# Computed tomography, conclusions

- Excellent spatial resolution
- 3D image
- Fast acquisition
- Weak soft tissue contrast (contrast agents available)
- Reconstruction algorithm
- Radiation dose