Algorithmic Game Theory - Introduction, Complexity

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About This Course

main goal of the course:

- dig deeper into game theory
- analyze the algorithmic and computational aspect of the problems in game theory
  - equilibrium computation algorithms (exact and approximate)
  - computational complexity (PLS, PPAD, FIXP, NP, $\Delta_2^P = P^{NP}$)
- extended foundations of algorithmic game theory
- main theorems, their impact, algorithms
- you

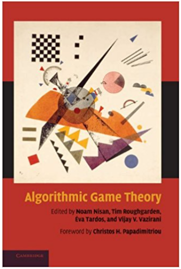
Grading: homework assignments (at least 2 correct out of 4) and presentation on a selected topic ($1/3$ of a research paper).

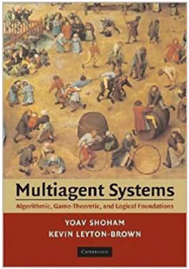
`https://cw.fel.cvut.cz/wiki/courses/xep36agt/lectures/start`

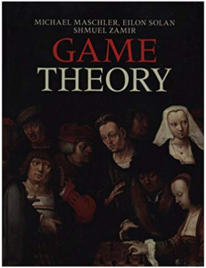
Books

There are 3 main books:

- Algorithmic Game Theory
  (by Noam Nisan and Tim Roughgarden)
- Multiagent Systems
  (by Yoav Shoham and Kevin Leyton-Brown)
- Game Theory
  (by Michael Maschler, Eilon Solan, Shmuel Zamir)







Outline of the Course

1. Introduction, Definitions [BB]
2. Nash's Theorem, Main Complexity Classes (PLS, PPAD, FIXP) [BB]
3. Computing and Approximating a Nash Equilibrium (Lemke Howson, MILP) [BB]
4. Computing Stackelberg Equilibria [BB]
5. Computing and Approximating Correlated Equilibria [BB]
6. Correlated Equilibrium in Succinct Games, Repeated Games [BB]
7. Multiarmed Bandit Problems [VL]
8. Learning in Normal-Form Games, Fictitious Play [VL]
9. Regret Matching, Counterfactual Regret Minimization [VL]
10. Continual Resolving in Extensive-Form Games (DeepStack) [VL]
11. Continuous Games 1 [TK]
12. Continuous Games 2 [TK]

Standard Representation of Games

standard normal-form representation – a game is a tuple $(\mathcal{N}, \mathcal{S}, u)$

- $\mathcal{N}$ is a set of players $i \in \mathcal{N} = \{1, \ldots, n\}$, $-i$ denotes all other players except i.
- $\mathcal{S}$ is a set of actions (pure strategies) $\mathcal{S} = \times_i \mathcal{S}_i$ (we often use $|\mathcal{S}_i| = m_i$)
- u_i is a utility function $u_i : \mathcal{S} \to \mathbb{R}$ (sometimes there is a cost function $c_i : \mathcal{S} \to \mathbb{R}, u_i(s) = -c_i(s)$)

also-known-as: strategic form, matrix form

we will refer to them as NFGs

in case of only two players: *bimatrix games*

Strategies

standard normal-form representation – a game is a tuple $(\mathcal{N}, \mathcal{S}, u)$

- *pure strategies* – $s_i \in \mathcal{S}_i$ (can be infinite)
- *mixed strategies* – probability distributions over pure strategies $\Delta(\mathcal{S}_i) = \left\{p^i \in \mathbb{R}^{|\mathcal{S}_i|} \mid \sum_{j=1}^{|\mathcal{S}_i|} p^i_j = 1 \wedge p^i_j \geq 0\right\}$, denoted σ
- *behavioral strategies* – vector of probability distributions over actions to play in each decision step
- *convex strategies* – arbitrary convex set $X \subseteq \mathbb{R}^{|\mathcal{S}|}$
- counting strategies, strategies with states, memory strategies, turing machine strategies

Beyond Standard Representation of Games

There are other representations that capture specific types of games more compactly compared to NFGs:

- *extensive-form games* – finite sequential games
  - represented as game trees (nodes are states, edges are actions, information sets connect indistinguishable states, utility values are in the leafs)
  - (but there are also standard Bayesian games, multi-agent influence diagrams (MAIDs), LIMIDs, . . .)
  - there are also less standard models for dynamic games (e.g., Normal-Form Games with Sequential Strategies [NFGSS])
- *stochastic games* – dynamic games with infinite/indefinite horizon
  - fully observable (generalization of repeated games), partially observable (one-sided, two-sided)

Beyond Standard Representation of Games (2)

- *congestion games* – abstract the network congestion games
  - We have n players, set of edges E, strategies for each player are *paths* in the network ($\mathcal{S}$), and there is a congestion function $c_e : \{0, 1, \ldots, n\} \rightarrow \mathbb{Z}^+$. When all players choose their strategy path $s_i \in \mathcal{S}_i$ we have the load of edge e, $\ell(e) = |\{s_i | e \in s_i\}|$ and $u_i = \sum_{e \in s_i} c_e(\ell(e))$
- *graphical games* – n-player games where the utility of one player typically depends only on few other players. They are represented as a graph, where agents are vertices and edge corresponds to the dependance between the two players. If the maximum degree of the graph is small ($d \ll n$), this representation offers exponentially smaller input $ns^{d+1} \ll ns^n$
- *action graph games* – even finer dependance than in graphical games based on actions

Beyond Standard Representation of Games (3)

- *polymatrix games* – specific graphical games, where we consider a bimatrix game for each edge (i.e., only pairwise interactions); quadratic size in ns
- *anonymous games*, *symmetric games*, ...

Continuous/Infinite Games

games over the unit square

- X, Y are set of "pure strategies" equal to interval $[0, 1]$
- we can reason about them similarly (although using calculus) to discrete games
- very useful in auctions, adversarial machine learning, any time you have a naturally infinite strategy space

Example: zero-sum game, $X = [0, 1]; Y = [0, 1]$, the payoff function is

$$u(x, y) = 4xy - 2x - y + 3, \qquad \forall x \in X,\ y \in Y$$

Why do we care?

One representation does not rule them all.

Depending on the representation we can get an exponential speed-up for specific types of problems.

Even if not, algorithms that work with compact representations can be a starting point if you are looking for an approximate solution to the original problem.

Solution Concepts

we want to find optimal strategies according to different notions of optimality:

- *maxmin strategies* – $\max_{s_i \in \mathcal{S}_i} \min_{s_{-i} \in \mathcal{S}_{-i}} u_i(s_i, s_{-i})$
- *minmax strategies* – $\min_{s_{-i} \in \mathcal{S}_{-i}} \max_{s_i \in \mathcal{S}_i} u_i(s_i, s_{-i})$
- can be defined for any type of strategies

if we seek *minmax strategies* over infinite sets, maximum or minimum over function $u_i(s_i, s_{-i})$ might not exist

- $\sup_{s_i \in \mathcal{S}_i} \inf_{s_{-i} \in \mathcal{S}_{-i}} u_i(s_i, s_{-i})$

$$\max_{s_i \in \mathcal{S}_i} \min_{s_{-i} \in \mathcal{S}_{-i}} u_i(s_i, s_{-i}) \leq \min_{s_{-i} \in \mathcal{S}_{-i}} \max_{s_i \in \mathcal{S}_i} u_i(s_i, s_{-i})$$

Solution Concepts (2)

stable solution concepts

- *best response* – let σ_{-i} be a strategy of players $-i$, $\max_{s_i \in \mathcal{S}_i} u_i(s_i, \sigma_{-i})$
  - we can define pure, mixed, behavioral best response
  - it is not always true that pure best responses are sufficient
  - $BR_i(\sigma_{-i})$ is a set of all best responses
- *Nash Equilibrium* – a strategy profile σ where every player is playing the best response to the strategies of other players; $\sigma_i \in BR_i(\sigma_{-i})$
- *(Strong) Stackelberg Equilibrium* – a strategy profile σ that maximizes the expected utility of player 1 (*leader*) where all other players (*followers*) are playing Nash Equilibrium;

$$\arg\max_{\sigma; \forall i \in \mathcal{N} \setminus \{1\}, \sigma_i \in BR_i(\sigma_{-i})} u_1(\sigma)$$

Solution Concepts (3)

- *Correlated Equilibrium* – a probability distribution over pure strategy profiles $p = \Delta(\mathcal{S})$ that recommends each player i to play the best response; $\forall s_i, s_i' \in \mathcal{S}_i$:

$$\sum_{s_{-i} \in \mathcal{S}_{-i}} p(s_i, s_{-i}) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in \mathcal{S}_{-i}} p(s_i, s_{-i}) u_i(s_i', s_{-i})$$

- *Coarse Correlated Equilibrium* – a probability distribution over pure strategy profiles $p = \Delta(\mathcal{S})$ that **in expectation** recommends each player i to play the best response; $\forall s_i \in \mathcal{S}_i$:

$$\sum_{s' \in \mathcal{S}'} p(s') u_i(s') \geq \sum_{s' \in \mathcal{S}'} p(s') u_i(s_i, s_{-i}')$$

- *Quantal Response Equilibrium* – modeling bounded rationality

$$p_j^i = \frac{\exp(u_i(s_j, \sigma_{-i}))}{\sum_{s_j' \in \mathcal{S}_i} \exp(u_i(s_j', \sigma_{-i}))}$$

Assumptions on Utilities

we can restrict to games with a specific utility function

- *zero-sum games* – meaningful for two-player games, where $u_1(s_1, s_2) = -u_2(s_1, s_2)$
- *almost zero-sum games* – games where there is an additional cost for one player $u_1(s_1, s_2) = -u_2(s_1, s_2) - c'(s_1)$
- *strategically zero-sum games* – let $A, B \in \mathbb{R}^{m_1 \times m_2}$ be the matrices of a bimatrix game. A game is SZS iff there exist $\alpha, \beta > 0$ and $D \in \mathbb{R}^{m_1 \times m_2}$ such that

$$\begin{aligned} \alpha A &= D + [\mathbf{b}^T, \mathbf{b}^T, \ldots, \mathbf{b}^T]^T \\ \beta B &= -D + [\mathbf{a}, \mathbf{a}, \ldots, \mathbf{a}] \end{aligned}$$

  for some $\mathbf{a} \in \mathbb{R}^{m_1}, \mathbf{b} \in \mathbb{R}^{m_2}$.
- *security games*, ...

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(besides the books)

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