CZECH TECHNICAL UNIVERSITY IN PRAGUE

Faculty of Electrical Engineering
Department of Cybernetics

A0M33EOA
Multi-objective Evolutionary Algorithms

Petr Pošík

Czech Technical University in Prague
Faculty of Electrical Engineering
Department of Cybernetics

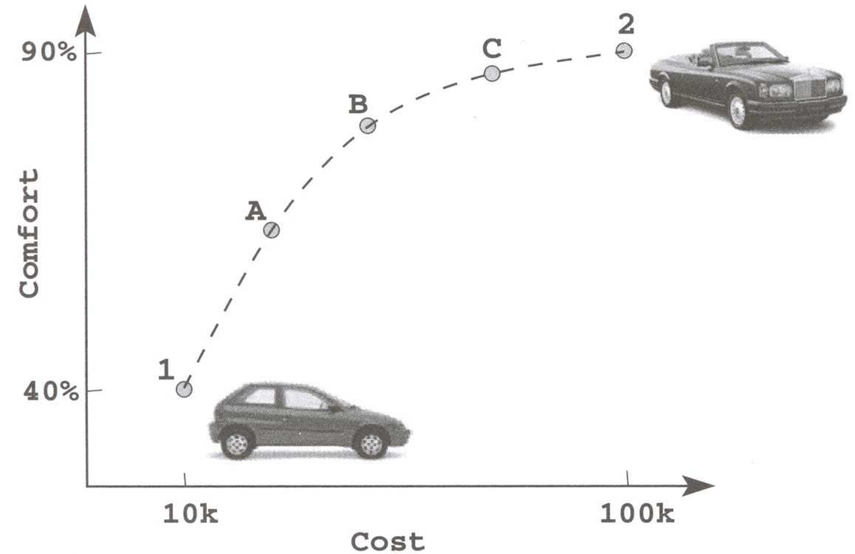
Heavilly using slides from **Jiří Kubalík**, CIIRC CTU, with permission.

Multi-objective Optimization

Multi-objective Optimization

Many real-world problems involve multiple objectives.

- **Conflicting objectives**
  - A solution that is extreme with respect to one objective requires a compromise in other objectives.
  - A sacrifice in one objective is related to the gain in other objective(s).
- Illustrative example: Buying a car
  - two extreme hypothetical cars 1 and 2,
  - cars with a trade-off between cost and comfort – A, B, and C.

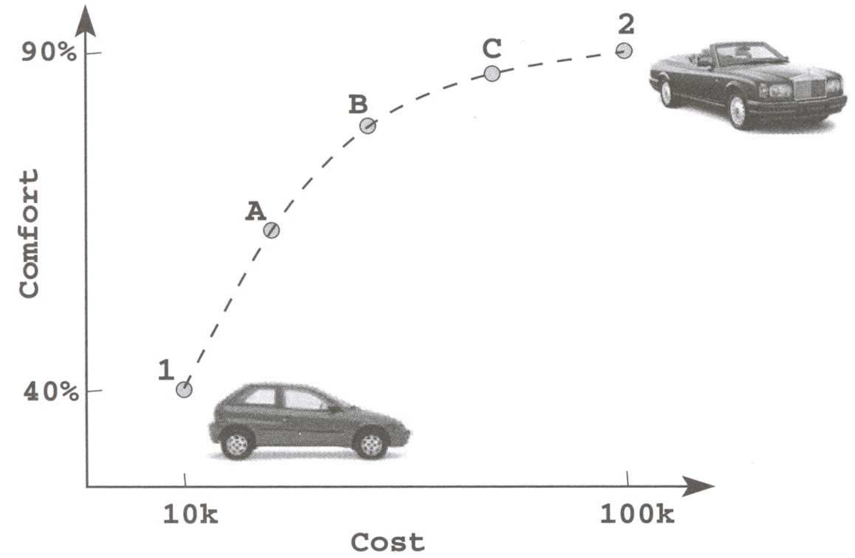


©Kalyanmoy Deb: Multi-Objective Optimization using Evolutionary Algorithms.

Multi-objective Optimization

Many real-world problems involve multiple objectives.

- **Conflicting objectives**
  - A solution that is extreme with respect to one objective requires a compromise in other objectives.
  - A sacrifice in one objective is related to the gain in other objective(s).
- Illustrative example: Buying a car
  - two extreme hypothetical cars 1 and 2,
  - cars with a trade-off between cost and comfort – A, B, and C.



©Kalyanmoy Deb: Multi-Objective Optimization using Evolutionary Algorithms.

Which solution out of all of the trade-off solutions **is the best** with respect to all objectives?

- Without any further information those trade-offs are indistinguishable.
- **A number of optimal solutions is sought in multiobjective optimization!**

Multi-Objective Optimization: Definition

Multi-objective Opt.
- MOO
- MOO Definition
- Dec./Obj. Space
- Example: Cantilever
- No Conflict
- Dominance
- MOO Properties
- MOO Goals
- Weighted Sum
- ε-Constraint
- Difficulties

Multi-objective EAs

Performance Measures

Summary

General form of multi-objective optimization problem

$$
\begin{array}{rlr}
\text{Minimize/maximize} & f_m(x), & m = 1, 2, \ldots, M; \\
\text{subject to} & g_j(x) \geq 0, & j = 1, 2, \ldots, J; \\
 & h_k(x) = 0, & k = 1, 2, \ldots, K; \\
 & x_i^{(L)} \leq x_i \leq x_i^{(U)}, & i = 1, 2, \ldots, n.
\end{array}
$$

- $\boldsymbol{x}$ is a vector of n decision variables: $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$.
- **Decision space** is constituted by variable bounds that restrict the value of each variable x_i to take a value within a lower $x_i^{(L)}$ and an upper $x_i^{(U)}$ bound.
- Inequality and equality constraints g_j and h_k.
- A solution x that satisfies all constraints and variable bounds is a **feasible solution**, otherwise it is called an **infeasible solution**.
- **Feasible space** is a set of all feasible solutions.
- Objective functions $f(\boldsymbol{x}) = (f_1(x), f_2(x), \ldots, f_M(x))$ constitute a multi-dimensional **objective space**.

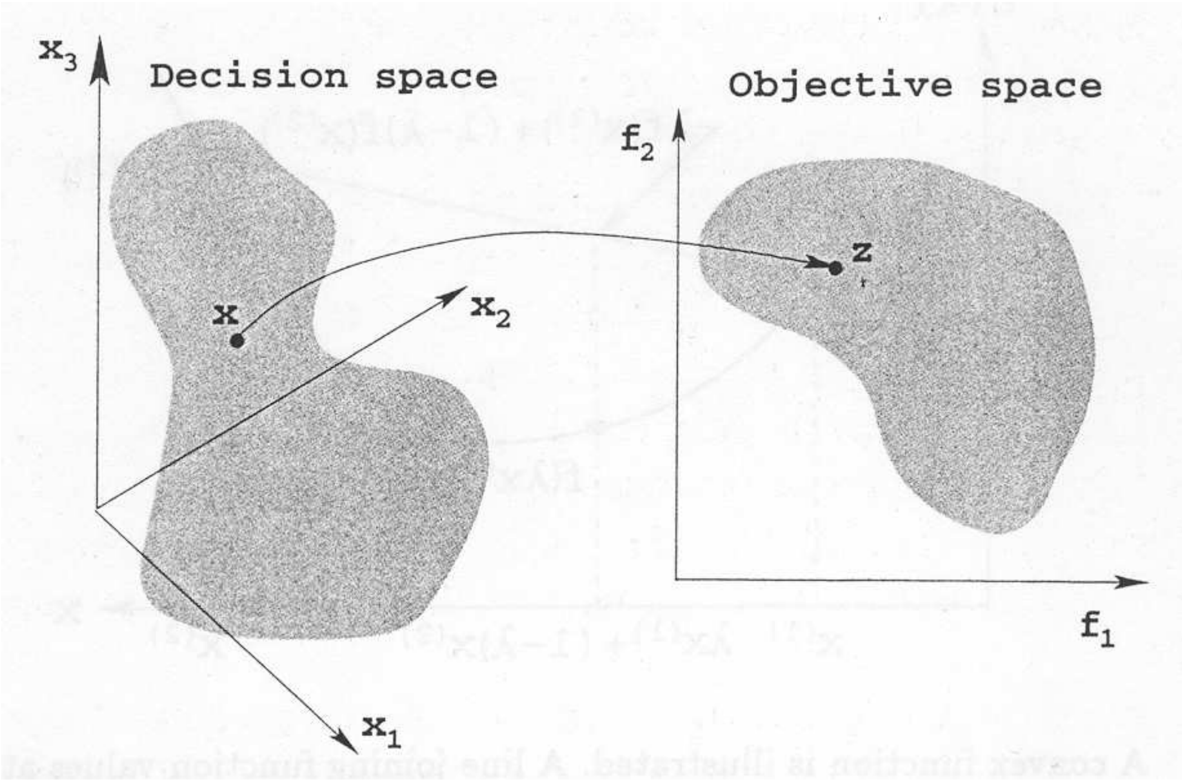
Multi-objective Opt.
- MOO
- MOO Definition
- Dec./Obj. Space
- Example: Cantilever
- No Conflict
- Dominance
- MOO Properties
- MOO Goals
- Weighted Sum
- ε-Constraint
- Difficulties

Multi-objective EAs

Performance Measures

Summary

Decision and Objective Space



©Kalyanmoy Deb: Multi-Objective Optimization using Evolutionary Algorithms.

- For each solution x in the decision space, there exists a point in the objective space

$$f(\boldsymbol{x}) = \boldsymbol{z} = (z_1, z_2, ..., z_M)^T$$

Multi-objective Opt.
• MOO
• MOO Definition
• Dec./Obj. Space
• Example: Cantilever
• No Conflict
• Dominance
• MOO Properties
• MOO Goals
• Weighted Sum
• ε-Constraint
• Difficulties
Multi-objective EAs
Performance Measures
Summary

Motivation Example: Cantilever Design Problem

Task: design a beam, defined by two *decision variables*,

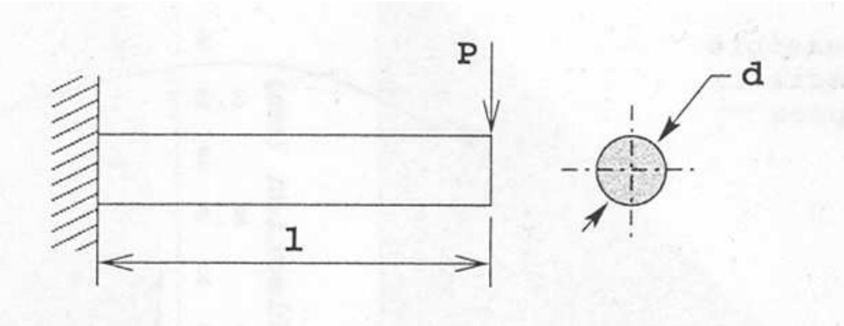
- diameter d and
- length l,

that can carry an end load P and is optimal with respect to *objectives*

- f_1: cantilever weight (to be minimized),
- f_2: endpoint deflection (to be minimized),

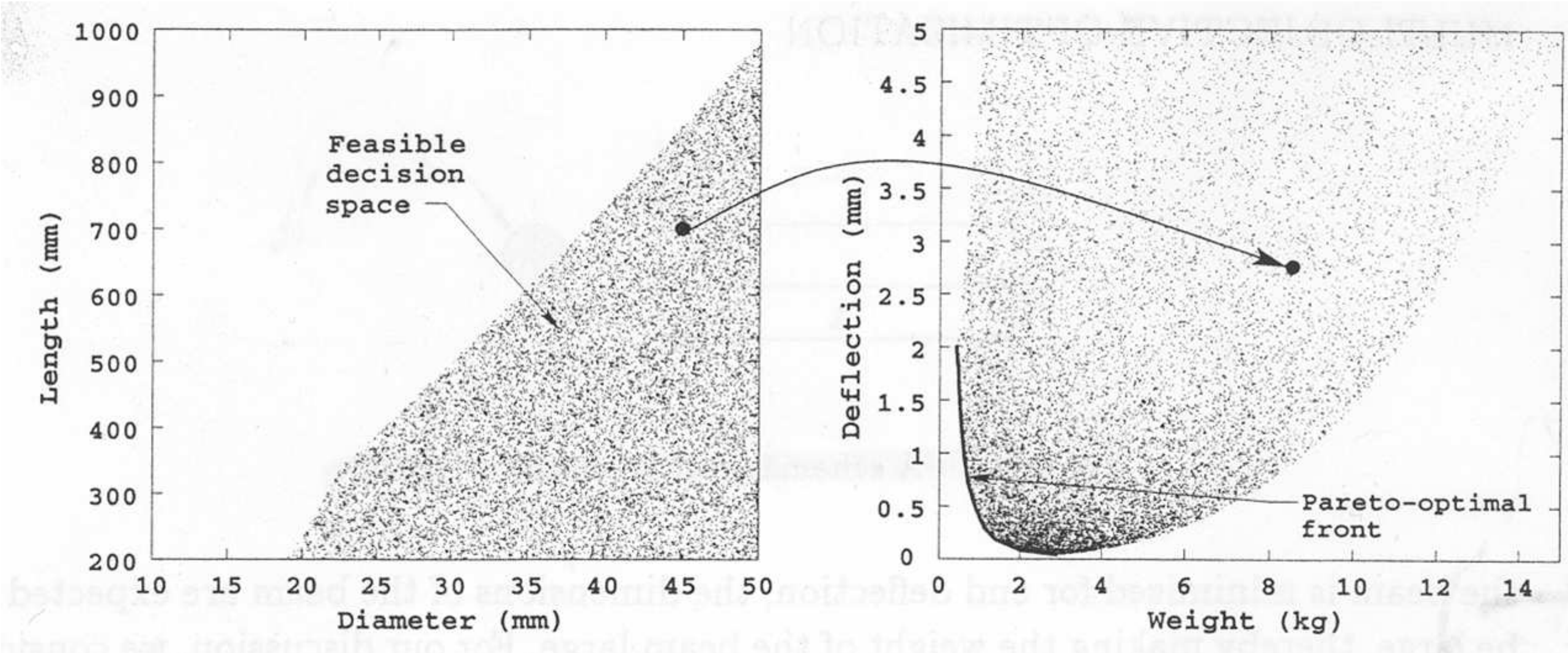
subject to the *constraints* that

- the developed maximum stress σ_{max} is less than the allowable stress S,
- the end deflection δ is smaller than a specified limit δ_{max}.



©Kalyanmoy Deb: Multi-Objective Optimization using Evolutionary Algorithms.

Cantilever Design Problem: Decision and Objective Space



©Kalyanmoy Deb: Multi-Objective Optimization using Evolutionary Algorithms.

Non-Conflicting Objectives

Multi-objective Opt.
- MOO
- MOO Definition
- Dec./Obj. Space
- Example: Cantilever
- No Conflict
- Dominance
- MOO Properties
- MOO Goals
- Weighted Sum
- ε-Constraint
- Difficulties

Multi-objective EAs

Performance Measures

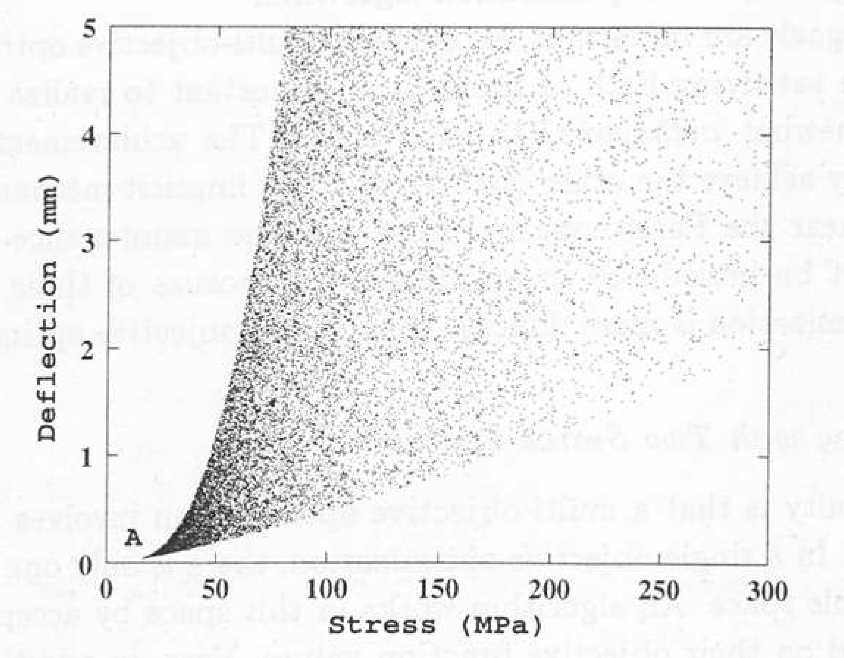
Summary

Existence of multiple trade-off solutions:

- Only if the objectives are in conflict with each other.
- If this does not hold then the cardinality of the Pareto-optimal set is one. (The optimum solutions w.r.t. individual objectives are the same.)

Example: Cantilever beam design problem:

- f_1: the end deflection δ (to be minimized),
- f_2: the maximum developed stress in the beam σ_{max} (to be minimized).



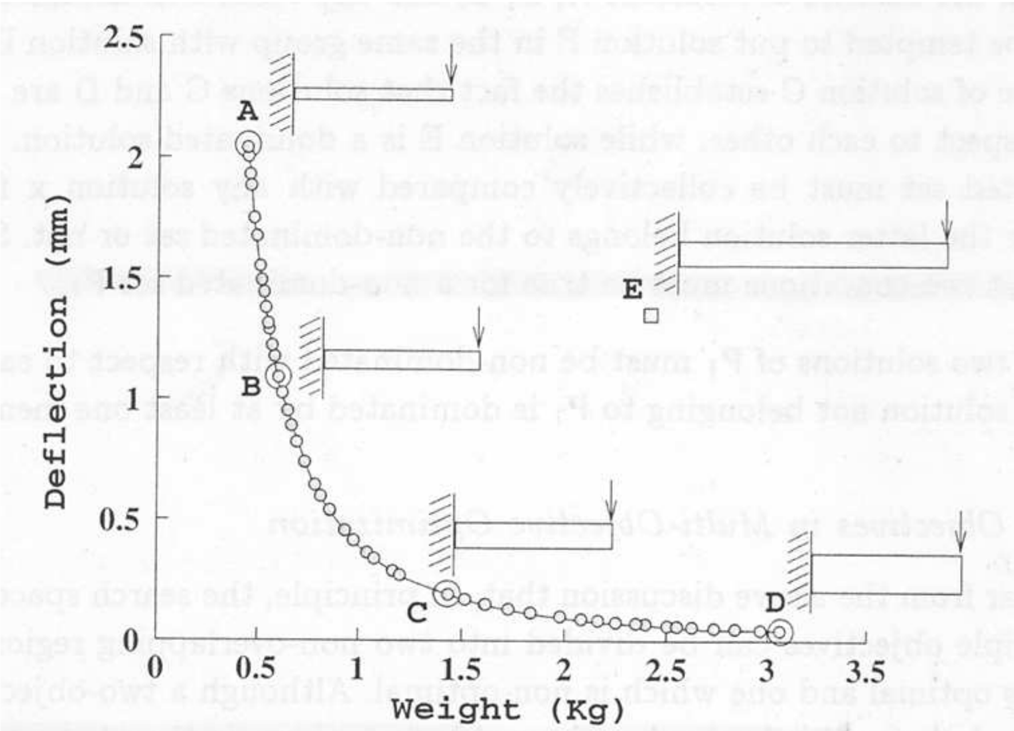
Dominance and Pareto-Optimal Solutions

Multi-objective Opt.
- MOO
- MOO Definition
- Dec./Obj. Space
- Example: Cantilever
- No Conflict
- Dominance
- MOO Properties
- MOO Goals
- Weighted Sum
- ε-Constraint
- Difficulties

Multi-objective EAs

Performance Measures

Summary



©Kalyanmoy Deb: Multi-Objective Optimization using Evolutionary Algorithms.

Domination: A solution $x^{(1)}$ is said to dominate another solution $x^{(2)}$, $x^{(1)} \preceq x^{(2)}$, if $x^{(1)}$ is not worse than $x^{(2)}$ in all objectives and $x^{(1)}$ is strictly better than $x^{(2)}$ in at least one objective.

Solutions A, B, C, D are *non-dominated* solutions (Pareto-optimal solutions)

Solution E is *dominated* by C and B (E is non-optimal).

Multi-objective Opt.
- MOO
- MOO Definition
- Dec./Obj. Space
- Example: Cantilever
- No Conflict
- Dominance
- MOO Properties
- MOO Goals
- Weighted Sum
- ε-Constraint
- Difficulties

Multi-objective EAs

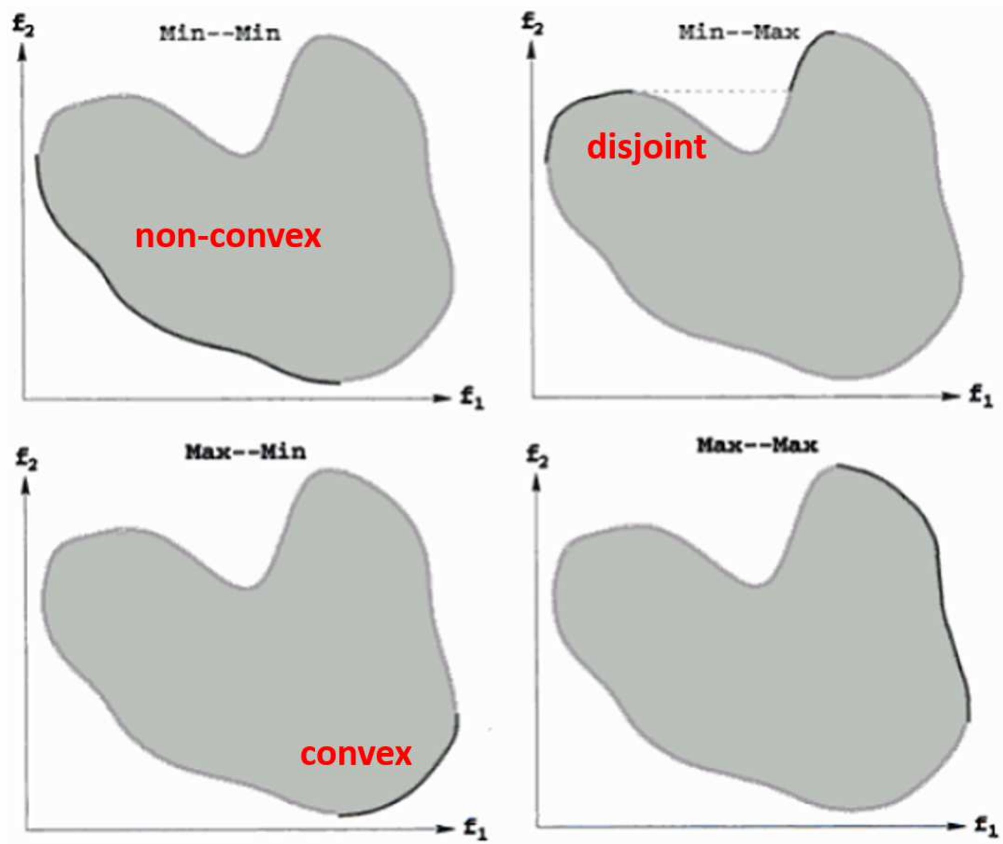
Performance Measures

Summary

Properties of Dominance-Based Multi-Objective Optimization

Non-dominated set: Among a set of solutions P, the non-dominated set of solutions P' are those that are not dominated by any member of the set P.

Globally Pareto-optimal set is the non-dominated set of the entire feasible space.



©Kalyanmoy Deb: Multi-Objective Optimization using Evolutionary Algorithms.

Goals of Dominance-Based Multi-Objective Optimization

Multi-objective Opt.
- MOO
- MOO Definition
- Dec./Obj. Space
- Example: Cantilever
- No Conflict
- Dominance
- MOO Properties
- MOO Goals
- Weighted Sum
- ε-Constraint
- Difficulties

Multi-objective EAs

Performance Measures

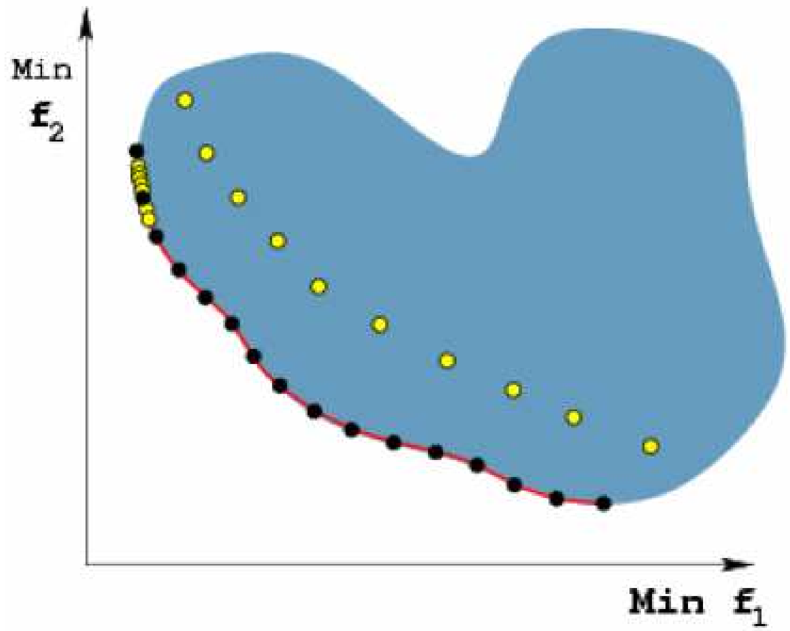
Summary

Every finite set of solutions P can be divided into two non-overlapping sets:

- **non-dominated set** P_1: contains all solutions that do not dominate each other
- **dominated set** P_2: any solution from P_2 is dominated by at least one solution from P_1

In the absence of other factors (e.g. preference for certain objectives, or for a particular region of the tradeoff surface) there are **two goals of multi-objective optimization**:

- **Quality:** Find a set of solutions as close as possible to the Pareto-optimal front.
- **Spread:** Find a set of non-dominated solutions as diverse as possible.



Classical Approaches: Weighted Sum Method

Multi-objective Opt.
- MOO
- MOO Definition
- Dec./Obj. Space
- Example: Cantilever
- No Conflict
- Dominance
- MOO Properties
- MOO Goals
- Weighted Sum
- ε-Constraint
- Difficulties

Multi-objective EAs

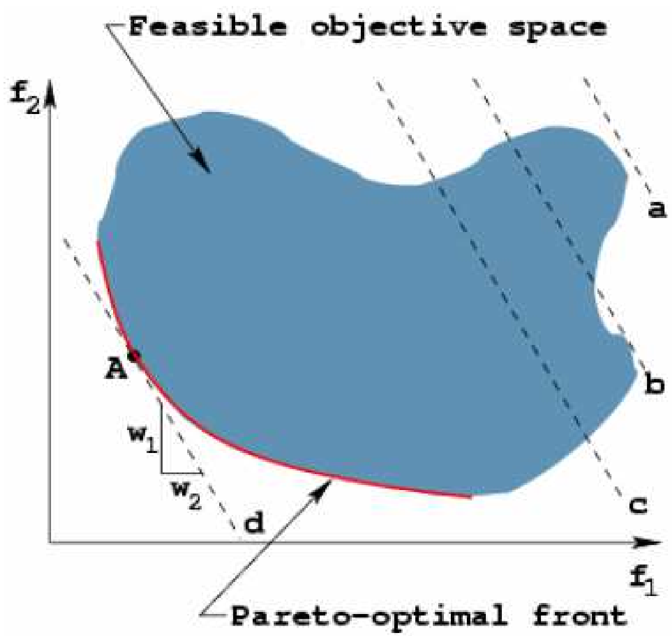
Performance Measures

Summary

- Construct a weighted sum of objectives and optimize

$$F(x) = \sum_{i=1}^{m} w_i \cdot f_i(x).$$

- User supplies weight vector w.
- Selection of weights w defines the slope of the line, which in turn determines the particular solution(s) on the boundary of the feasible space.



Difficulties with Weighted Sum Method

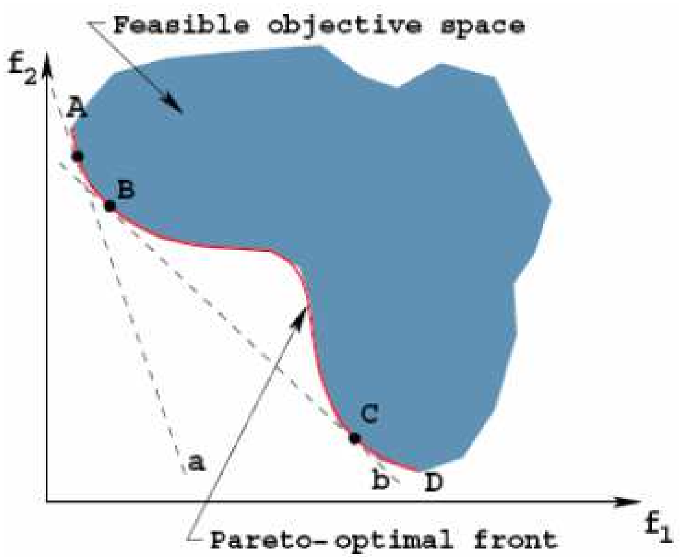
Multi-objective Opt.
- MOO
- MOO Definition
- Dec./Obj. Space
- Example: Cantilever
- No Conflict
- Dominance
- MOO Properties
- MOO Goals
- Weighted Sum
- ε-Constraint
- Difficulties

Multi-objective EAs

Performance Measures

Summary

- Need to know weight vector w.
- To find a set of trade-off solutions, the method must be run many times with varying w.
- Non-uniformity in Pareto-optimal solutions.
- Inability to find some Pareto-optimal solutions (in non-convex region).
- However, a solution of this approach is always Pareto-optimal.



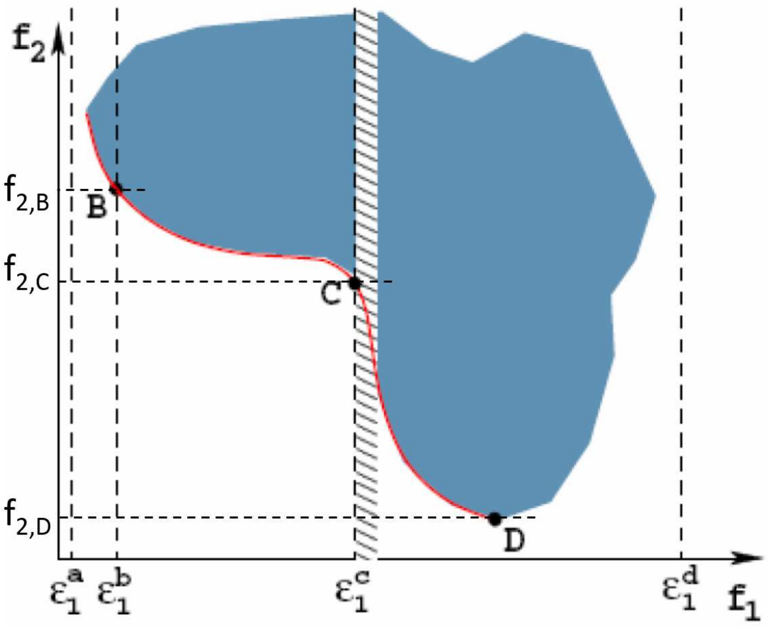
Classical Approaches: ε-Constraint Method

Method: Minimize a primary objective while expressing all the other objectives in the form of inequality constraints

$$\text{minimize } f_p(x)$$
$$\text{subject to } f_i(x) \le \varepsilon_i, \text{ for } i = 1, \ldots, m, i \ne p.$$

Example:

$$\text{minimize } f_2(x)$$
$$\text{subject to } f_1(x) \le \varepsilon_1.$$



Remarks:

- To find a whole set of trade-off solutions, the method must be run many times.
- Need to know relevant ε vectors to ensure a feasible solution.
- Non-uniformity in Pareto-optimal solutions.
- However, any Pareto-optimal solution can be found with this method.

Difficulties with Most Classical Approaches

Multi-objective Opt.
- MOO
- MOO Definition
- Dec./Obj. Space
- Example: Cantilever
- No Conflict
- Dominance
- MOO Properties
- MOO Goals
- Weighted Sum
- ε-Constraint
- Difficulties

Multi-objective EAs

Performance Measures

Summary

- Need to run a single-objective optimizer many times.
- A lot of problem knowledge is required.
- Even then, good distribution of solutions is not guaranteed.
- Multi-objective optimization as an application of single-objective optimization.

Multi-objective EAs

Why and How Use EAs for Multi-Objective Optimization?

Why?

- *Population approach* suits well to find multiple solutions.
- *Niche-preservation methods* can be exploited to find diverse solutions.
- *Implicit parallelism* helps provide a parallel search.
  Multiple applications of classical methods do not constitute a parallel search.

How?

- Modify the *fitness computation*.
- Emphasize non-dominated solutions for *convergence*.
- Emphasize unique solutions for *diversity*.

Multi-Objective Evolutionary Algorithms

Multi-objective Opt.

Multi-objective EAs
- MOEAs: Why?
- MOEAs
- NSGA
- Fitness Sharing
- FS: Example
- NSGA: Fitness
- NSGA: Conclusions
- NSGA-II
- NSGA-II: Diversity
- NSGA-II Steps
- NSGA vs NSGA-II
- Simulation results
- Constraints
- NSGA-II Sym. Reg.
- SPEA2
- SPEA2 Steps
- SPEA2 Fitness
- SPEA2 Diversity
- SPEA2 Replacement
- SPEA2: Conclusions

Performance Measures

Summary

- **Pareto Archived Evolution Strategy** (PAES)
  Knowles, J.D., Corne, D.W. (2000) Approximating the nondominated front using the Pareto archived evolution strategy. Evolutionary Computation, 8(2), pp. 149-172
- **Multiple Objective Genetic Algorithm** (MOGA)
  Carlos M. Fonseca, Peter J. Fleming: Genetic Algorithms for Multiobjective Optimization: Formulation, Discussion and Generalization, In Genetic Algorithms: Proceedings of the Fifth International Conference, 1993
- **Niched-Pareto Genetic Algorithm** (NPGA)
  Jeffrey Horn, Nicholas Nafpliotis, David E. Goldberg: A Niched Pareto Genetic Algorithm for Multiobjective Optimization, Proceedings of the First IEEE Conference on Evolutionary Computation, IEEE World Congress on Computational Intelligence, 1994
- **SPEA2**
  Zitzler, E., Laumanns, M., Thiele, L.: SPEA2: Improving the Strength Pareto Evolutionary Algorithm For Multiobjective Optimization, In: Evolutionary Methods for Design, Optimisation, and Control, Barcelona, Spain, pp. 19-26, 2002
- **NSGA**
  Srinivas, N., and Deb, K.: Multi-objective function optimization using non-dominated sorting genetic algorithms, Evolutionary Computation Journal 2(3), pp. 221-248, 1994
- **NSGA-II**
  Kalyanmoy Deb, Samir Agrawal, Amrit Pratap, and T Meyarivan: A Fast Elitist Non-Dominated Sorting Genetic Algorithm for Multi-Objective Optimization: NSGA-II, In Proceedings of the Parallel Problem Solving from Nature VI Conference, 2000
- . . .

Multi-objective Opt.
Multi-objective EAs
- MOEAs: Why?
- MOEAs
- NSGA
- Fitness Sharing
- FS: Example
- NSGA: Fitness
- NSGA: Conclusions
- NSGA-II
- NSGA-II: Diversity
- NSGA-II Steps
- NSGA vs NSGA-II
- Simulation results
- Constraints
- NSGA-II Sym. Reg.
- SPEA2
- SPEA2 Steps
- SPEA2 Fitness
- SPEA2 Diversity
- SPEA2 Replacement
- SPEA2: Conclusions

Performance Measures
Summary

Non-Dominated Sorting Genetic Algorithm (NSGA)

Common features with the standard GA:

- variation operators – crossover and mutation,
- selection method – Stochastic Reminder Roulette-Wheel,
- standard generational evolutionary model.

Differences of NSGA from SGA:

- fitness assignment scheme which *prefers non-dominated solutions*, and
- fitness sharing strategy which *preserves diversity among solutions of each non-dominated front*.

NSGA steps:

1. Initialize population of solutions.
2. Repeat
   - Calculate objective values and assign fitness values.
   - Generate new population.

   Until stopping condition is fulfilled.

Fitness Sharing

Multi-objective Opt.

Multi-objective EAs
- MOEAs: Why?
- MOEAs
- NSGA
- Fitness Sharing
- FS: Example
- NSGA: Fitness
- NSGA: Conclusions
- NSGA-II
- NSGA-II: Diversity
- NSGA-II Steps
- NSGA vs NSGA-II
- Simulation results
- Constraints
- NSGA-II Sym. Reg.
- SPEA2
- SPEA2 Steps
- SPEA2 Fitness
- SPEA2 Diversity
- SPEA2 Replacement
- SPEA2: Conclusions

Performance Measures

Summary

Diversity preservation method originally proposed for solving multi-modal optimization problems so that GA is able to discover and evenly sample all optima.

Idea: decrease fitness of similar solutions

Algorithm to calculate the shared fitness value of i-th individual in population of size N

1. Calculate the distances d_{ij} of individual i to all individuals j.
2. Calculate values of *sharing function* between individual i and all individuals j:

$$Sh(d_{ij}) = \begin{cases} 1 - \left(\frac{d_{ij}}{\sigma_{share}}\right)^{\alpha}, \text{if } d_{ij} \leq \sigma_{share}, \\ 0, \text{otherwise.} \end{cases}$$

3. Calculate *niche count* nc_i of individual i:

$$nc_i = \sum_{j=1}^{N} Sh(d_{ij})$$

4. Calculate *shared fitness* of individual i:

$$f_i' = f_i / nc_i$$

Remark: If $d = 0$, then $Sh(d) = 1$, meaning that two solutions are identical. If $d \geq \sigma_{share}$, then $Sh(d) = 0$ meaning that two solutions do not have any sharing effect on each other.

Fitness Sharing: Example

Multi-objective Opt.

Multi-objective EAs
- MOEAs: Why?
- MOEAs
- NSGA
- Fitness Sharing
- FS: Example
- NSGA: Fitness
- NSGA: Conclusions
- NSGA-II
- NSGA-II: Diversity
- NSGA-II Steps
- NSGA vs NSGA-II
- Simulation results
- Constraints
- NSGA-II Sym. Reg.
- SPEA2
- SPEA2 Steps
- SPEA2 Fitness
- SPEA2 Diversity
- SPEA2 Replacement
- SPEA2: Conclusions

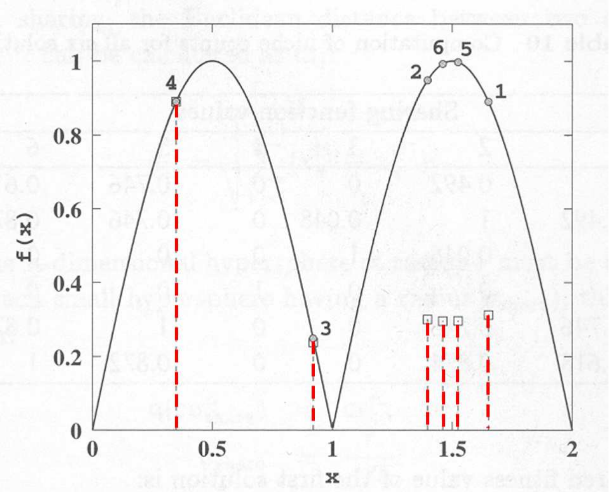
Performance Measures

Summary

Bimodal function, six solutions, and corresponding shared fitness values.

- $\sigma_{share} = 0.5$, $\alpha = 1$.

Sol. i	String	Decoded value	$x^{(i)}$	f_i	nc_i	f'_i
1	110100	52	1.651	0.890	2.856	0.312
2	101100	44	1.397	0.948	3.160	0.300
3	011101	29	0.921	0.246	1.048	0.235
4	001011	11	0.349	0.890	1.000	0.890
5	110000	48	1.524	0.997	3.364	0.296
6	101110	46	1.460	0.992	3.364	0.295



Let's take the first solution:

- $d_{11} = 0.0$, $d_{12} = 0.254$, $d_{13} = 0.731$, $d_{14} = 1.302$, $d_{15} = 0.127$, $d_{16} = 0.191$
- $Sh(d_{11}) = 1$, $Sh(d_{12}) = 0.492$, $Sh(d_{13}) = 0$, $Sh(d_{14}) = 0$, $Sh(d_{15}) = 0.746$, $Sh(d_{16}) = 0.618$.
- $nc_1 = 1 + 0.492 + 0 + 0 + 0.746 + 0.618 = 2.856$
- $f'(1) = f(1)/nc_1 = 0.890/2.856 = 0.312$

Remark:

- The above example computes d_{ij} in decision space, $d_{ij} = d(\mathbf{x}_i - \mathbf{x}_j)$.
- To create diverse set of non-dominated solutions, we have to compute it in the objective space, e.g., $d_{ij} = d(f(\mathbf{x}_i) - f(\mathbf{x}_j)) = d(\mathbf{z}_i - \mathbf{z}_j)$ (or see next slide).

NSGA: Fitness Assignment

Multi-objective Opt.
Multi-objective EAs
- MOEAs: Why?
- MOEAs
- NSGA
- Fitness Sharing
- FS: Example
- NSGA: Fitness
- NSGA: Conclusions
- NSGA-II
- NSGA-II: Diversity
- NSGA-II Steps
- NSGA vs NSGA-II
- Simulation results
- Constraints
- NSGA-II Sym. Reg.
- SPEA2
- SPEA2 Steps
- SPEA2 Fitness
- SPEA2 Diversity
- SPEA2 Replacement
- SPEA2: Conclusions

Performance Measures
Summary

Input: Set P of solutions with assigned objective values.
Output: Set of solutions with assigned fitness values (the bigger the better).

1. Choose sharing parameter σ_{share}, small positive number ϵ, initialize $f_{max} = PopSize$ and front counter $front = 1$
2. Find set $P' \subset P$ of non-dominated solutions.
3. For each $q \in P'$,
   - assign fitness $f(q) = f_{max}$,
   - calculate sharing function with all solutions in P', niche count nc_q among solutions of P' only, the normalized Euclidean distance d_{ij} is calculated as

$$d_{ij} = \sqrt{\sum_{m=1}^{M} \left(\frac{f_m^{(i)} - f_m^{(j)}}{f_m^{\max} - f_m^{\min}} \right)^2},$$

   - calculate shared fitness $f'(q) = f(q)/nc_q$.
4. $f_{max} = min(f'(q) : q \in P') - \epsilon$,
   $P = P \setminus P'$,
   $front = front + 1$.
5. If not all solutions are assessed go to step 2, otherwise stop.

NSGA: Fitness Assignment (cont.)

Multi-objective Opt.

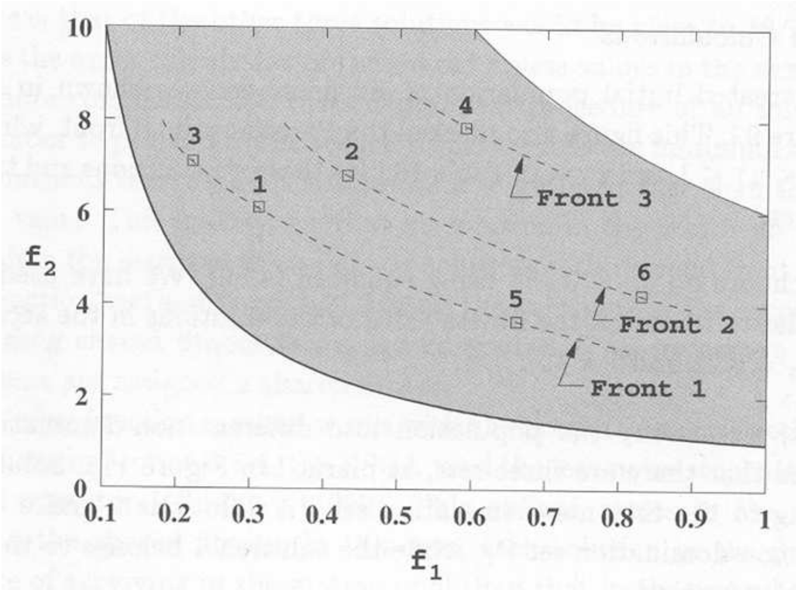
Multi-objective EAs
- MOEAs: Why?
- MOEAs
- NSGA
- Fitness Sharing
- FS: Example
- NSGA: Fitness
- NSGA: Conclusions
- NSGA-II
- NSGA-II: Diversity
- NSGA-II Steps
- NSGA vs NSGA-II
- Simulation results
- Constraints
- NSGA-II Sym. Reg.
- SPEA2
- SPEA2 Steps
- SPEA2 Fitness
- SPEA2 Diversity
- SPEA2 Replacement
- SPEA2: Conclusions

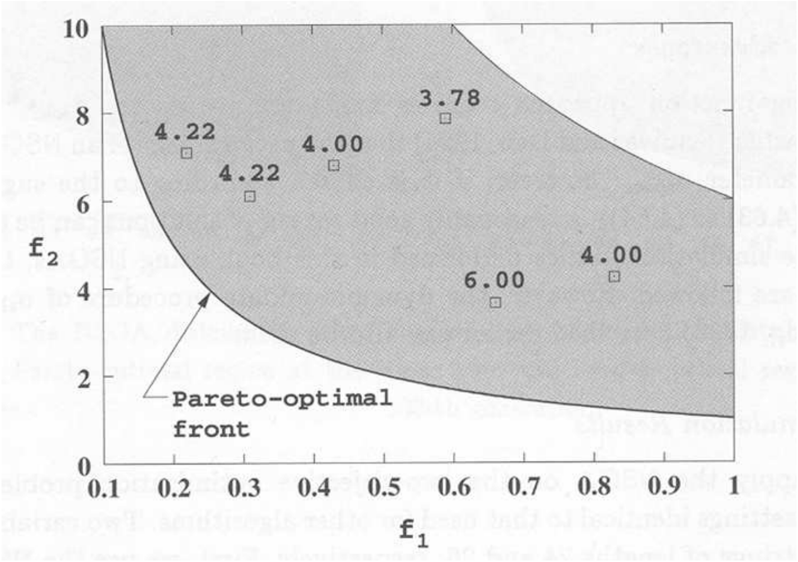
Performance Measures

Summary

Example:

- First, 6 solutions are classified into different non-dominated fronts.
- Then, the fitness values are calculated according to the fitness sharing method.
  - The sharing function method is used front-wise.
  - Within a front, less dense solutions have better fitness values.





©Kalyanmoy Deb: Multi-Objective Optimization using Evolutionary Algorithms.

NSGA: Conclusions

Multi-objective Opt.

Multi-objective EAs
- MOEAs: Why?
- MOEAs
- NSGA
- Fitness Sharing
- FS: Example
- NSGA: Fitness
- NSGA: Conclusions
- NSGA-II
- NSGA-II: Diversity
- NSGA-II Steps
- NSGA vs NSGA-II
- Simulation results
- Constraints
- NSGA-II Sym. Reg.
- SPEA2
- SPEA2 Steps
- SPEA2 Fitness
- SPEA2 Diversity
- SPEA2 Replacement
- SPEA2: Conclusions

Performance Measures

Summary

Computational complexity

- Governed by the non-dominated sorting procedure and the sharing function implementation.
  - **non-dominated sorting** – complexity of $O(MN^3)$.
  - **sharing function** – requires every solution in a front to be compared with every other solution in the same front, total of $\sum_{j=1}^{\rho} |P_j|^2$, where ρ is a number of fronts.
    Each distance computation requires evaluation of n differences between parameter values.
    In the worst case, when $\rho = 1$, the overall complexity is of $O(nN^2)$.

Advantages:

- Assignment of fitness according to non-dominated sets makes the algorithm converge towards the Pareto-optimal region.
- Sharing allows phenotypically diverse solutions to emerge.

Disadvantages:

- non-elitist
- sensitive to the sharing method parameter σ_{share}
  - requires some guidelines for setting the σ_{share}
  - e.g., $\sigma_{share} = \frac{0.5}{\sqrt[n]{q}}$ based on the expected number of optima q

Multi-objective Opt.

Multi-objective EAs
• MOEAs: Why?
• MOEAs
• NSGA
• Fitness Sharing
• FS: Example
• NSGA: Fitness
• NSGA: Conclusions
• NSGA-II
• NSGA-II: Diversity
• NSGA-II Steps
• NSGA vs NSGA-II
• Simulation results
• Constraints
• NSGA-II Sym. Reg.
• SPEA2
• SPEA2 Steps
• SPEA2 Fitness
• SPEA2 Diversity
• SPEA2 Replacement
• SPEA2: Conclusions

Performance Measures

Summary

NSGA-II

Fast non-dominated sorting approach

- Computational complexity is $O(MN^2)$.

Diversity preservation

- The sharing function method is replaced with a **crowded comparison approach**.
- Parameterless approach.

Elitist evolutionary model

- Only the best solutions survive to subsequent generations.

NSGA-II: Diversity preservation

Multi-objective Opt.

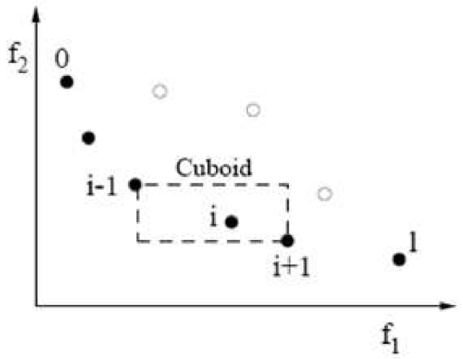
Multi-objective EAs
- MOEAs: Why?
- MOEAs
- NSGA
- Fitness Sharing
- FS: Example
- NSGA: Fitness
- NSGA: Conclusions
- NSGA-II
- NSGA-II: Diversity
- NSGA-II Steps
- NSGA vs NSGA-II
- Simulation results
- Constraints
- NSGA-II Sym. Reg.
- SPEA2
- SPEA2 Steps
- SPEA2 Fitness
- SPEA2 Diversity
- SPEA2 Replacement
- SPEA2: Conclusions

Performance Measures

Summary

Density estimation: **crowding distance** estimates how much unique the solution is.

- For individual i, find its predecessor and successor in each objective.
- Crowding distance $i^{distance}$ is the sum of differences in objective values of predecessor and successor across all objectives.
- For individuals with extreme value of at least one objective, $i^{distance} = \infty$.



©Kalyanmoy Deb: Multi-Objective Optimization using Evolutionary Algorithms.

Crowded comparison operator $\prec_c$:

- Every solution in the population has two attributes:
  1. non-domination rank i^{rank}, and
  2. crowding distance $i^{distance}$
- A partial order $\prec_c$ is defined as:

$$i \prec_c j \quad \text{if} \quad i^{rank} < j^{rank} \text{ or } (i^{rank} = j^{rank} \text{ and } i^{distance} > j^{distance}).$$

NSGA-II: Evolutionary Model

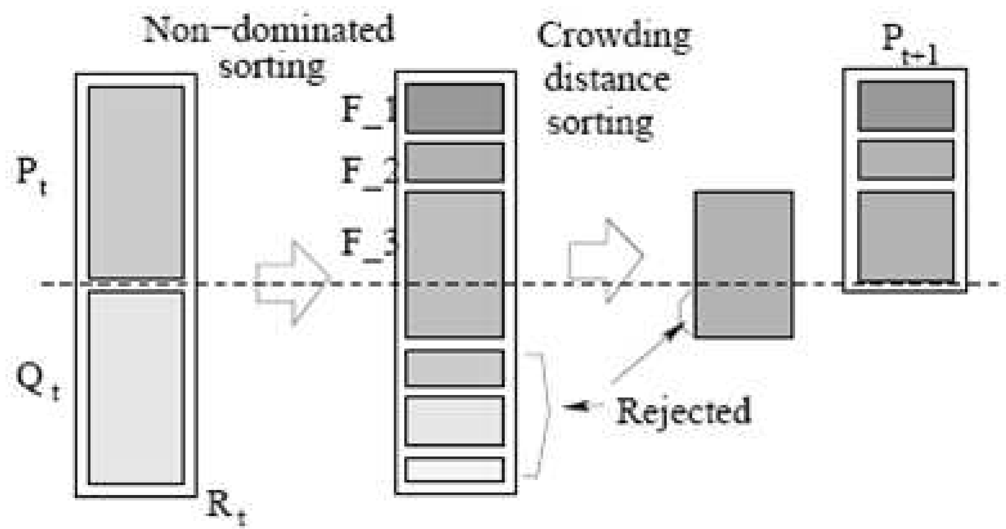
Multi-objective Opt.

Multi-objective EAs
- MOEAs: Why?
- MOEAs
- NSGA
- Fitness Sharing
- FS: Example
- NSGA: Fitness
- NSGA: Conclusions
- NSGA-II
- NSGA-II: Diversity
- NSGA-II Steps
- NSGA vs NSGA-II
- Simulation results
- Constraints
- NSGA-II Sym. Reg.
- SPEA2
- SPEA2 Steps
- SPEA2 Fitness
- SPEA2 Diversity
- SPEA2 Replacement
- SPEA2: Conclusions

Performance Measures

Summary

1. Sort the current population P_t based on the non-domination.
   Each solution is assigned a fitness equal to its non-domination level (1 is the best).
2. Apply the usual binary tournament selection, recombination, and mutation to create a child population Q_t of size N.
3. Combine both populations: $R_t = P_t \cup Q_t$. (Steady-state algorithm, elitism is ensured.)
4. Perform replacement (environmental selection): Population P_{t+1} is formed according to the following schema

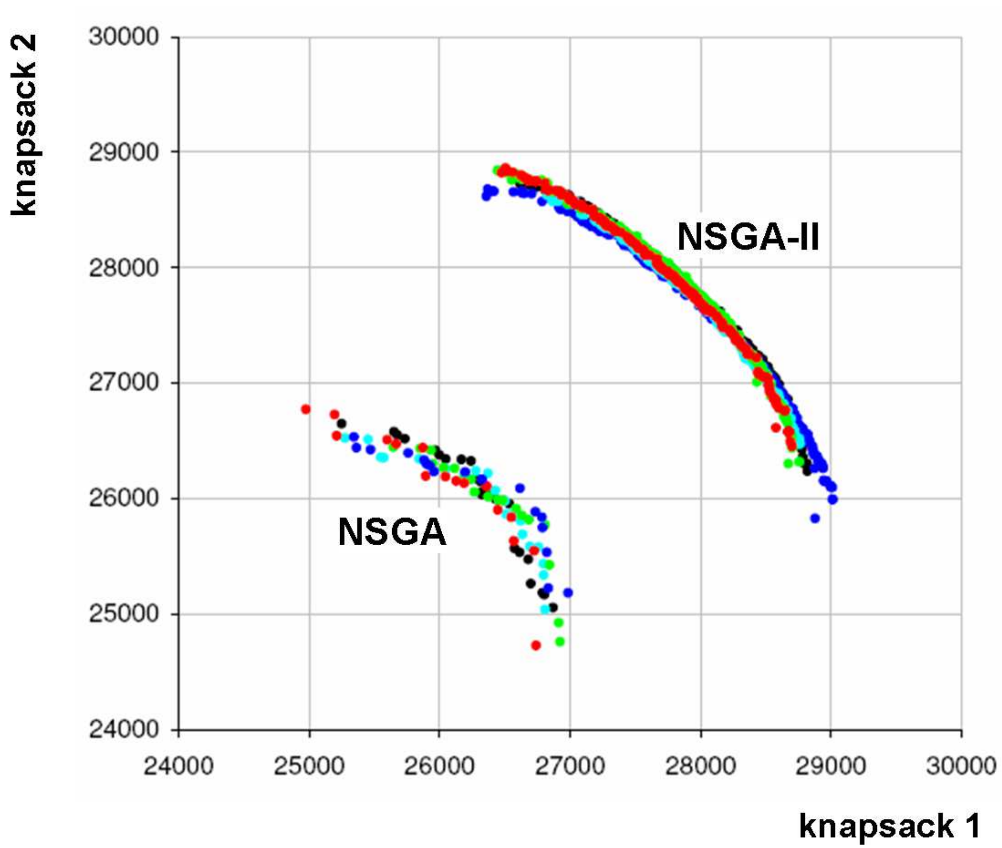


©Kalyanmoy Deb: Multi-Objective Optimization using Evolutionary Algorithms.

Simulation Results: NSGA vs. NSGA-II

Comparison of NSGA nad NSGA-II on bi-objective 0/1 Knapsack Problem with 750 items.

NSGA-II outperforms NSGA with respect to both performance measures.

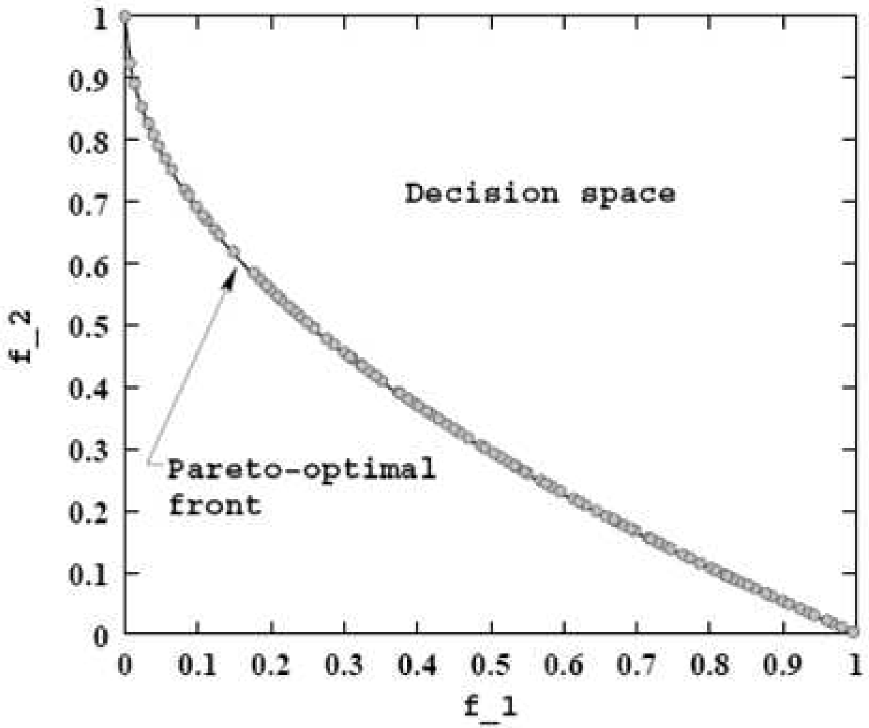


Multi-objective Opt.
Multi-objective EAs
- MOEAs: Why?
- MOEAs
- NSGA
- Fitness Sharing
- FS: Example
- NSGA: Fitness
- NSGA: Conclusions
- NSGA-II
- NSGA-II: Diversity
- NSGA-II Steps
- NSGA vs NSGA-II
- Simulation results
- Constraints
- NSGA-II Sym. Reg.
- SPEA2
- SPEA2 Steps
- SPEA2 Fitness
- SPEA2 Diversity
- SPEA2 Replacement
- SPEA2: Conclusions

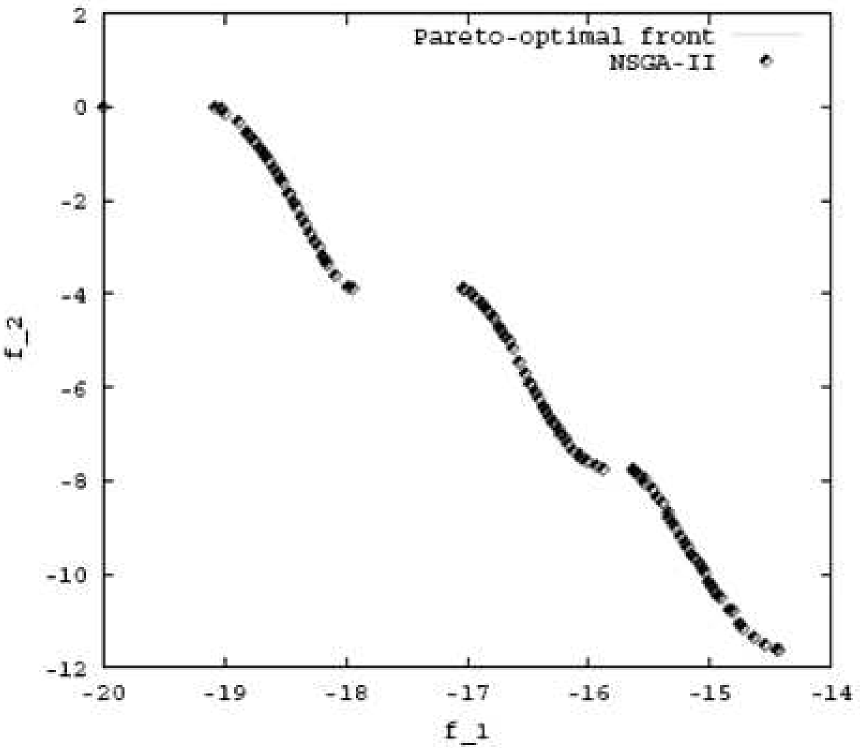
Performance Measures
Summary

NSGA-II: Simulation Results on Various Types of Problems

Problem with continuous Pareto-optimal front



Problem with discontinuous Pareto-optimal front



NSGA-II: Constraint Handling Approach

Multi-objective Opt.
Multi-objective EAs
• MOEAs: Why?
• MOEAs
• NSGA
• Fitness Sharing
• FS: Example
• NSGA: Fitness
• NSGA: Conclusions
• NSGA-II
• NSGA-II: Diversity
• NSGA-II Steps
• NSGA vs NSGA-II
• Simulation results
• Constraints
• NSGA-II Sym. Reg.
• SPEA2
• SPEA2 Steps
• SPEA2 Fitness
• SPEA2 Diversity
• SPEA2 Replacement
• SPEA2: Conclusions
Performance Measures
Summary

Binary tournament selection with modified domination concept is used to choose the better solution out of the two solutions i and j, randomly picked up from the population.

In the presence of constraints, each solution in the population can be either **feasible** or **infeasible**, so that there are the following three possible situations:

1. both solutions are feasible,
2. one is feasible and other is not,
3. both are infeasible.

Multi-objective Opt.
Multi-objective EAs
- MOEAs: Why?
- MOEAs
- NSGA
- Fitness Sharing
- FS: Example
- NSGA: Fitness
- NSGA: Conclusions
- NSGA-II
- NSGA-II: Diversity
- NSGA-II Steps
- NSGA vs NSGA-II
- Simulation results
- Constraints
- NSGA-II Sym. Reg.
- SPEA2
- SPEA2 Steps
- SPEA2 Fitness
- SPEA2 Diversity
- SPEA2 Replacement
- SPEA2: Conclusions

Performance Measures
Summary

NSGA-II: Constraint Handling Approach

Binary tournament selection with modified domination concept is used to choose the better solution out of the two solutions i and j, randomly picked up from the population.

In the presence of constraints, each solution in the population can be either **feasible** or **infeasible**, so that there are the following three possible situations:

1. both solutions are feasible,
2. one is feasible and other is not,
3. both are infeasible.

Constrained-domination: A solution i is said to constrained-dominate a solution j, if any of the following conditions is true:

1. Solution i is feasible and solution j is not.
2. Solutions i and j are both infeasible, but solution i has a smaller overall constraint violation.
3. Solutions i and j are feasible, and solution i dominates solution j.

NSGA-II: Simulation Results (cont.)

Multi-objective Opt.

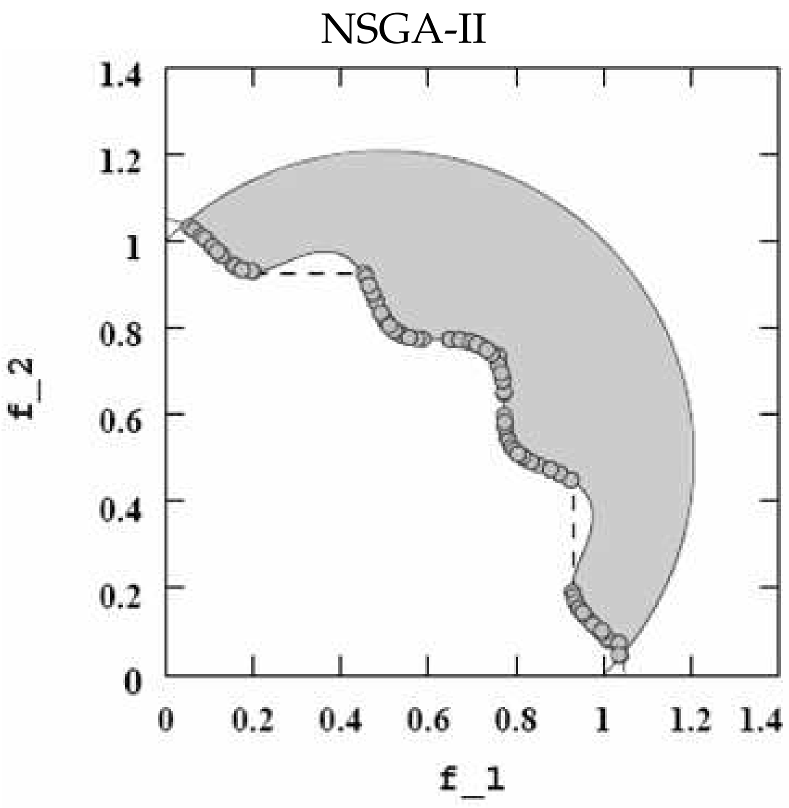
Multi-objective EAs
- MOEAs: Why?
- MOEAs
- NSGA
- Fitness Sharing
- FS: Example
- NSGA: Fitness
- NSGA: Conclusions
- NSGA-II
- NSGA-II: Diversity
- NSGA-II Steps
- NSGA vs NSGA-II
- Simulation results
- Constraints
- NSGA-II Sym. Reg.
- SPEA2
- SPEA2 Steps
- SPEA2 Fitness
- SPEA2 Diversity
- SPEA2 Replacement
- SPEA2: Conclusions

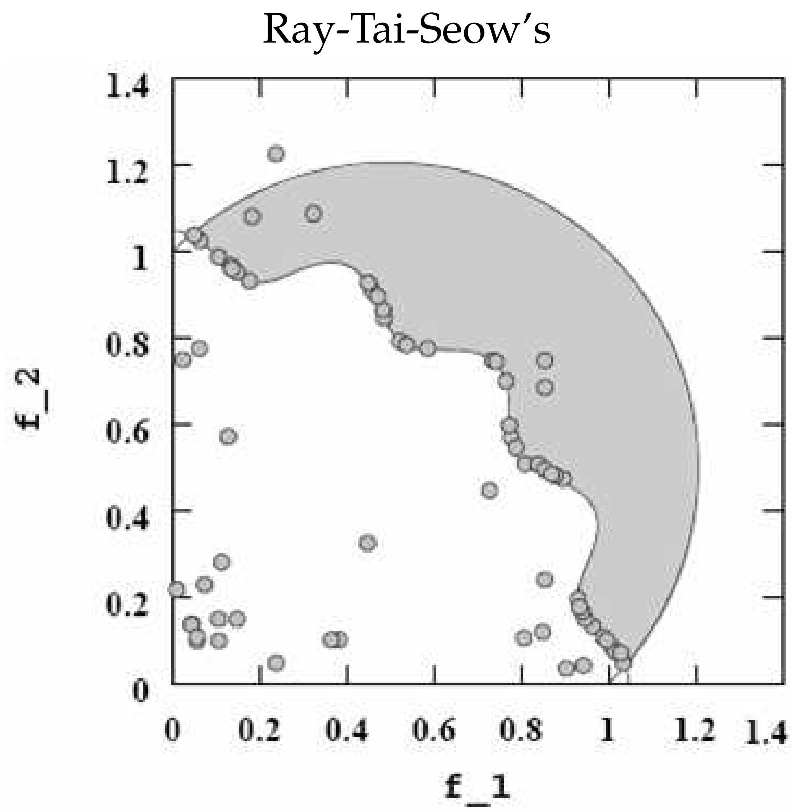
Performance Measures

Summary

Comparison of NSGA-II and Ray-Tai-Seow's Constraint handling approach

- Ray, T., Tai, K. and Seow, K.C. "Multiobjective Design Optimization by an Evolutionary Algorithm", Engineering Optimization, Vol.33, No.4, pp. 399-424, 2001.

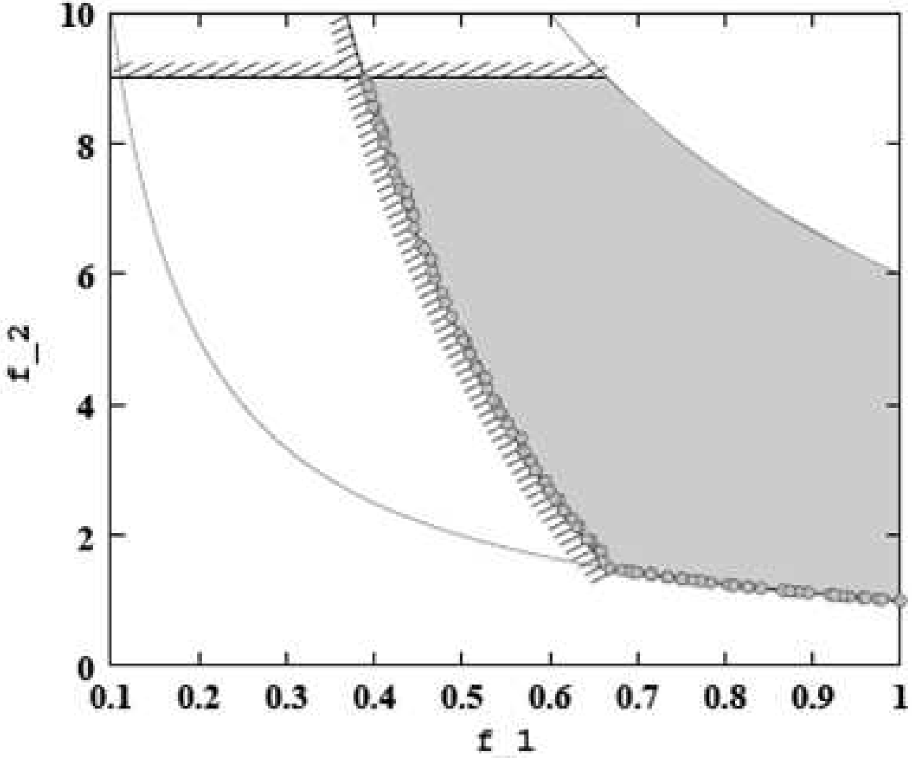




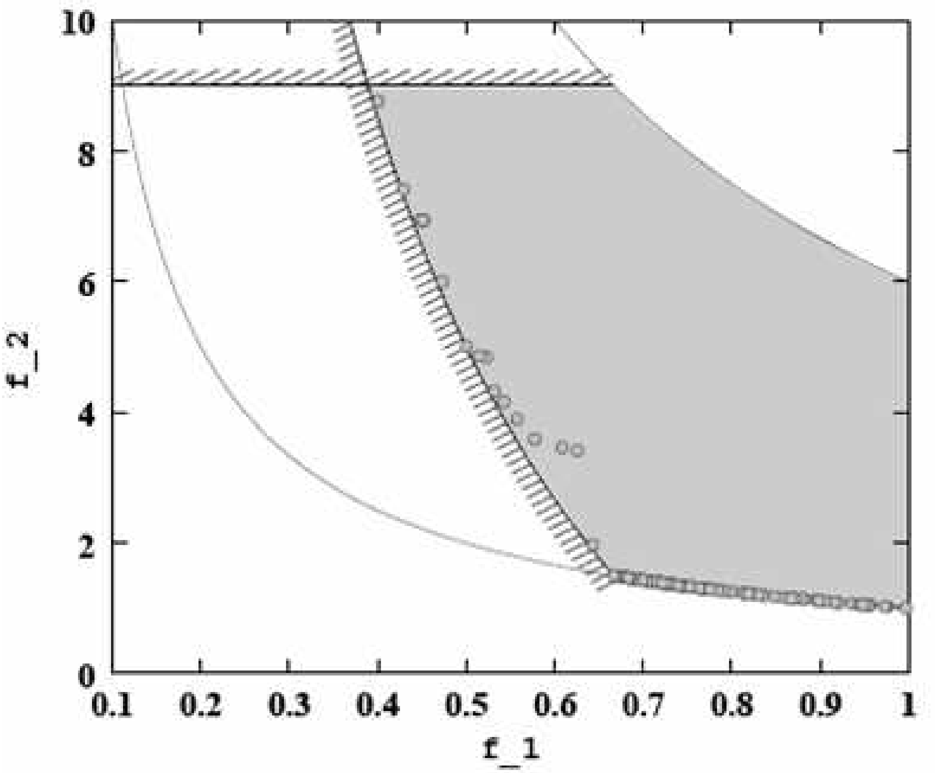
NSGA-II: Simulation Results (cont.)

Comparison of NSGA-II and Ray-Tai-Seow's's Constraint handling approach:

NSGA-II



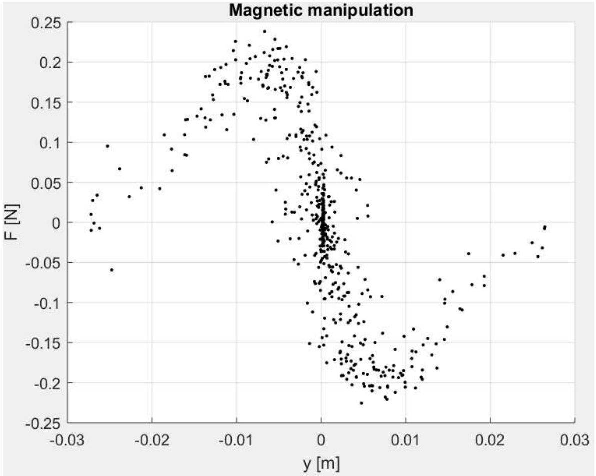
Ray-Tai-Seow's



NSGA-II: Bi-objective Symbolic Regression

Optimization objectives:

- Minimize MSE on the training data set.
- Minimize deviation of the symbolic models from the desired properties.

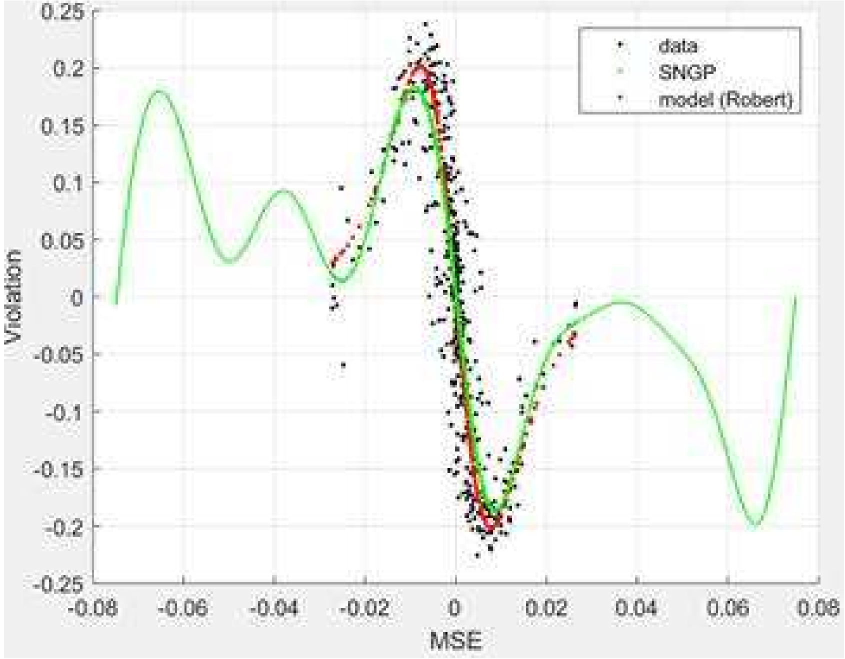


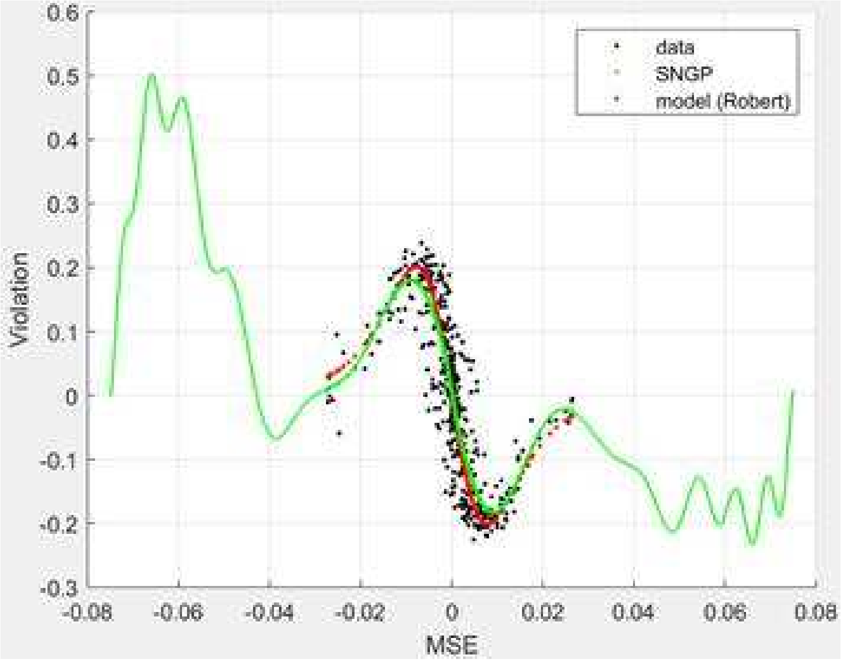
Desired properties:

- Monotonically increasing in the intervals $y = \langle -0.075, -0.01 \rangle$ and $y = \langle 0.01, 0.075 \rangle$
- Monotonically decreasing in the interval $y = \langle -0.07, 0.07 \rangle$
- $F(y) \geq 0$, for $y \in \langle -0.075, 0.0 \rangle$
- $F(y) \leq 0$, for $y \in \langle 0.0, 0.075 \rangle$
- $|F(0.0)| < 0.005$
- $|F(-0.075) - 0.001| < 0.0005$
- $|F(0.075) + 0.001| < 0.0005$

NSGA-II: Bi-objective Symbolic Regression

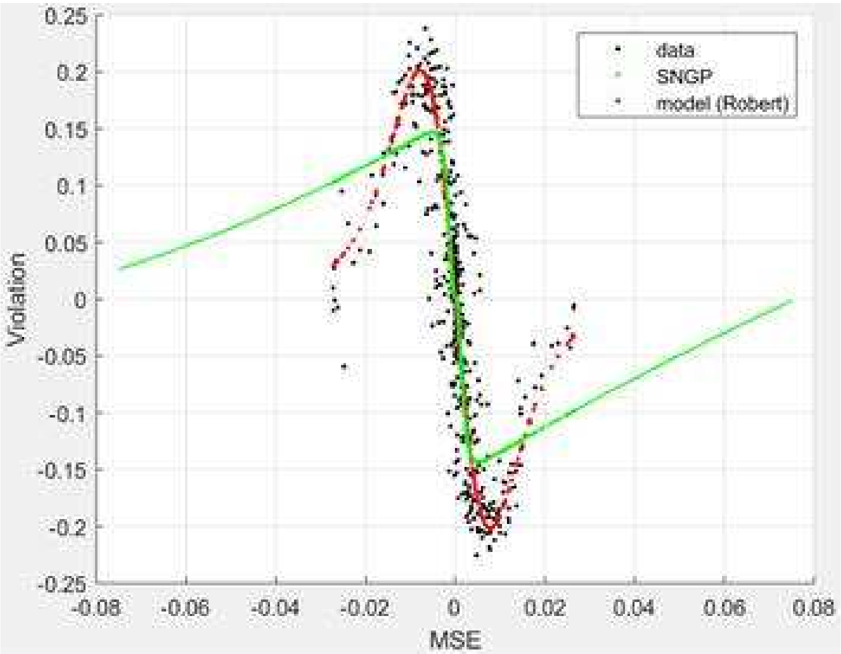
Well-fit models *w.r.t. the MSE on training data* only:

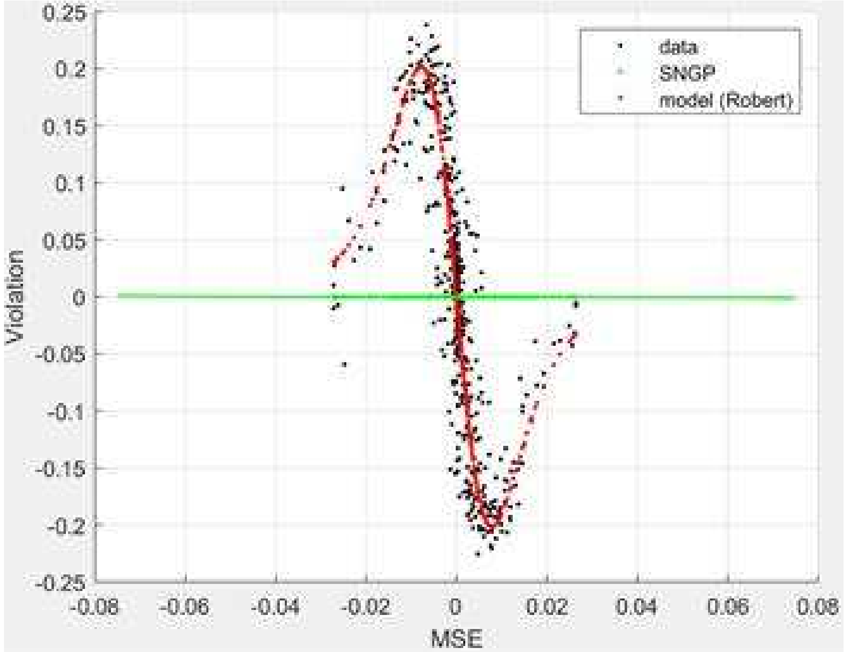




NSGA-II: Bi-objective Symbolic Regression

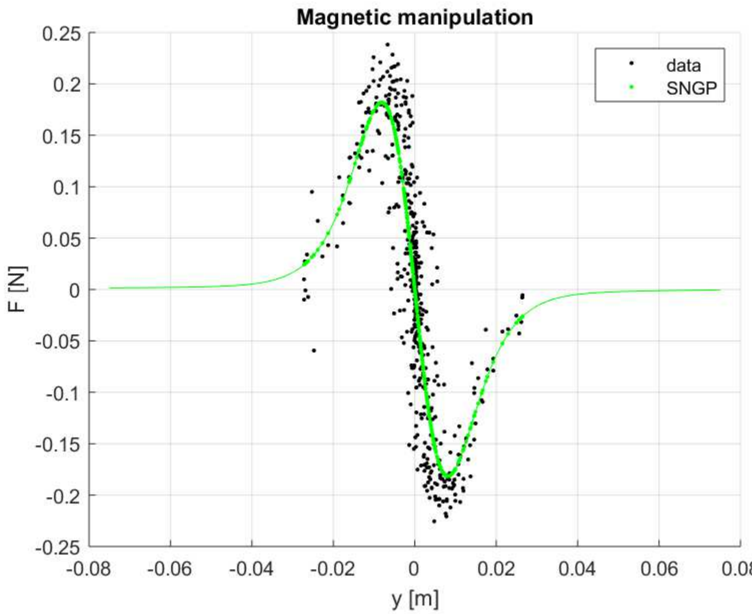
Well-fit models *w.r.t. the constraint violations*:

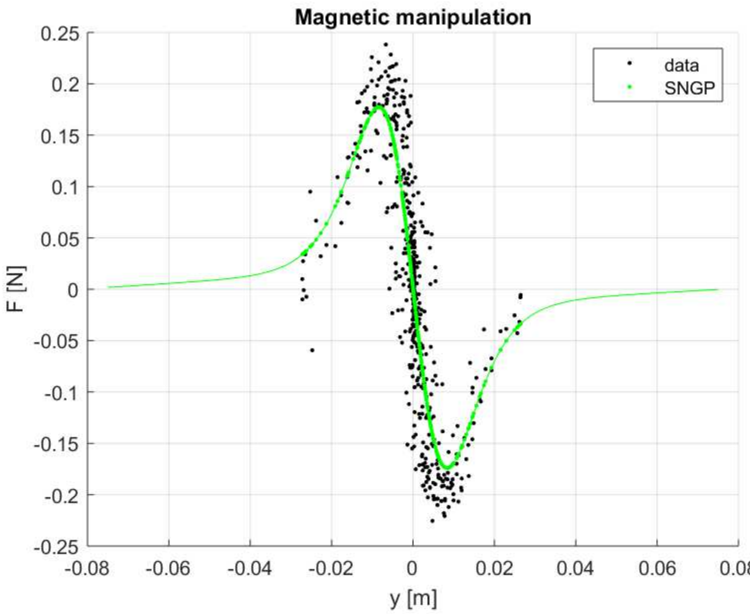




NSGA-II: Bi-objective Symbolic Regression

Models with *small MSE on training data* that *fully comply with the constraints*:

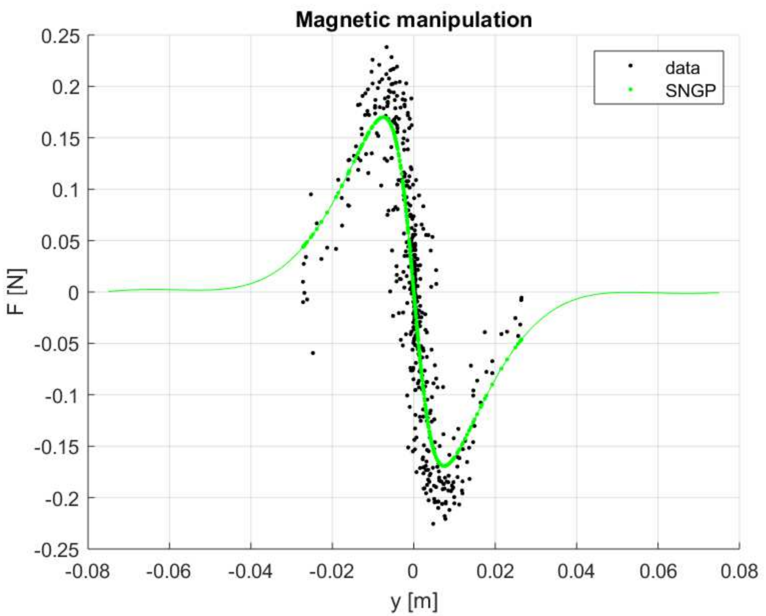




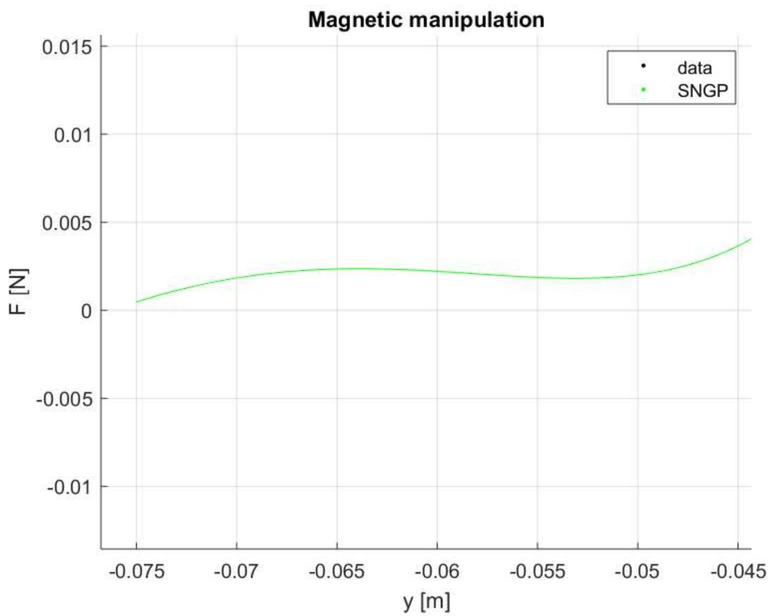
NSGA-II: Bi-objective Symbolic Regression

Models with *small MSE on training data* that *almost fully comply with the constraints*:

The whole model



Detail of left tail



Strength Pareto Evolutionary Algorithm 2 (SPEA2)

Multi-objective Opt.

Multi-objective EAs
- MOEAs: Why?
- MOEAs
- NSGA
- Fitness Sharing
- FS: Example
- NSGA: Fitness
- NSGA: Conclusions
- NSGA-II
- NSGA-II: Diversity
- NSGA-II Steps
- NSGA vs NSGA-II
- Simulation results
- Constraints
- NSGA-II Sym. Reg.
- SPEA2
- SPEA2 Steps
- SPEA2 Fitness
- SPEA2 Diversity
- SPEA2 Replacement
- SPEA2: Conclusions

Performance Measures

Summary

SPEA2 maintains two sets of solutions:

- **regular population** of newly generated solutions, and
- **archive**, which contains a representation of the nondominated front among all solutions considered so far.

Archive:

- **The archive size is fixed**, i.e., whenever the number of nondominated individuals is less than the predefined archive size, the archive is filled up by *good* dominated individuals.
- A **truncation method** is invoked when the nondominated front exceeds the archive limit.
- A member of the archive is only removed if
  1. a solution has been found that dominates it, or
  2. the maximum archive size is exceeded and the portion of the front where the archive member is located is overcrowded.
- The archive makes it possible not to lose certain portions of the current nondominated front due to random effects.
- **All individuals in the archive participate in selection.**

Multi-objective Opt.
Multi-objective EAs
• MOEAs: Why?
• MOEAs
• NSGA
• Fitness Sharing
• FS: Example
• NSGA: Fitness
• NSGA: Conclusions
• NSGA-II
• NSGA-II: Diversity
• NSGA-II Steps
• NSGA vs NSGA-II
• Simulation results
• Constraints
• NSGA-II Sym. Reg.
• SPEA2
• SPEA2 Steps
• SPEA2 Fitness
• SPEA2 Diversity
• SPEA2 Replacement
• SPEA2: Conclusions
Performance Measures
Summary

SPEA2: Algorithm

Input: N is the population size, $\overline{N}$ is the archive size.

1. **Initialization**: Generate an initial population P_0 and create the empty archive $\overline{P}_0 = \emptyset$. Set $t = 0$.
2. **Fitness assignment**: Calculate fitness of individuals in P_t and $\overline{P}_t$.
3. **Environmental selection**: Copy all nondominated individuals in P_t and $\overline{P}_t$ to $\overline{P}_{t+1}$.
   - If size of $\overline{P}_{t+1}$ exceeds $\overline{N}$ then reduce $\overline{P}_{t+1}$ using the truncation operator.
   - If size of $\overline{P}_{t+1}$ is less than $\overline{N}$ then fill $\overline{P}_{t+1}$ with dominated solutions in P_t and $\overline{P}_t$.
4. **Termination**: If $t \geq T$ then return nondominated solutions in $\overline{P}_{t+1}$. Stop.
5. **Mating selection**: Perform binary tournament selection with replacement on $\overline{P}_{t+1}$ in order to fill the mating pool.
6. **Variation**: Apply recombination and mutation operators to the mating pool and fill P_{t+1} with the generated solutions.
7. Increment generation counter $t = t + 1$.
8. Go to Step 2.

Multi-objective Opt.
Multi-objective EAs
• MOEAs: Why?
• MOEAs
• NSGA
• Fitness Sharing
• FS: Example
• NSGA: Fitness
• NSGA: Conclusions
• NSGA-II
• NSGA-II: Diversity
• NSGA-II Steps
• NSGA vs NSGA-II
• Simulation results
• Constraints
• NSGA-II Sym. Reg.
• SPEA2
• SPEA2 Steps
• SPEA2 Fitness
• SPEA2 Diversity
• SPEA2 Replacement
• SPEA2: Conclusions
Performance Measures
Summary

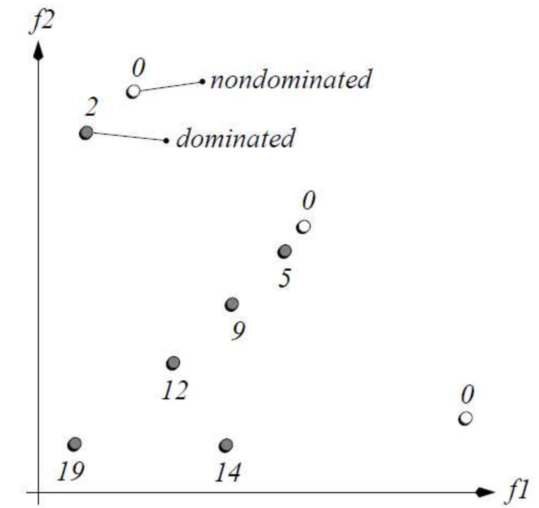
SPEA2: Fitness Assignment

Fitness assignment (fitness should be minimized):

- For each individual, both dominating and dominated solutions are taken into account.
- Each individual i in the archive $\overline{P}_t$ and in the population P_t is assigned a **strength value** $S(i)$, representing the number of solutions it dominates.
- The raw fitness $R(i)$ of an individual i is calculated as

$$R(i) = \sum_{j \in P_t + \overline{P}_t, j \succ i} S(j),$$

  i.e., $R(i)$ is determined by the strengths of its dominators in both archive and population. $R(i) = 0$ corresponds to a nondominated solution.
- Since the **raw fitness assignment** is based on the concept of Pareto dominance, it **may fail when most individuals do not dominate each other**.



Both objectives should be maximized.

SPEA2: Density Estimation

Multi-objective Opt.

Multi-objective EAs
- MOEAs: Why?
- MOEAs
- NSGA
- Fitness Sharing
- FS: Example
- NSGA: Fitness
- NSGA: Conclusions
- NSGA-II
- NSGA-II: Diversity
- NSGA-II Steps
- NSGA vs NSGA-II
- Simulation results
- Constraints
- NSGA-II Sym. Reg.
- SPEA2
- SPEA2 Steps
- SPEA2 Fitness
- SPEA2 Diversity
- SPEA2 Replacement
- SPEA2: Conclusions

Performance Measures

Summary

Density information is incorporated to discriminate between individuals having identical raw fitness values.

The density at any point is estimated as a (decreasing) function of the distance to the k-th nearest data point – calculated as the inverse of the distance to the k-th nearest neighbor.

- k equal to the square root of the sample size is used: $k = \sqrt{N + \overline{N}}$.
- **Density** $D(i)$ is calculated as

$$D(i) = \frac{1}{\sigma_i^k + 2}$$

where σ_i^k is the distance to the k-th nearest neighbor and it is made sure that $D(i) < 1$.

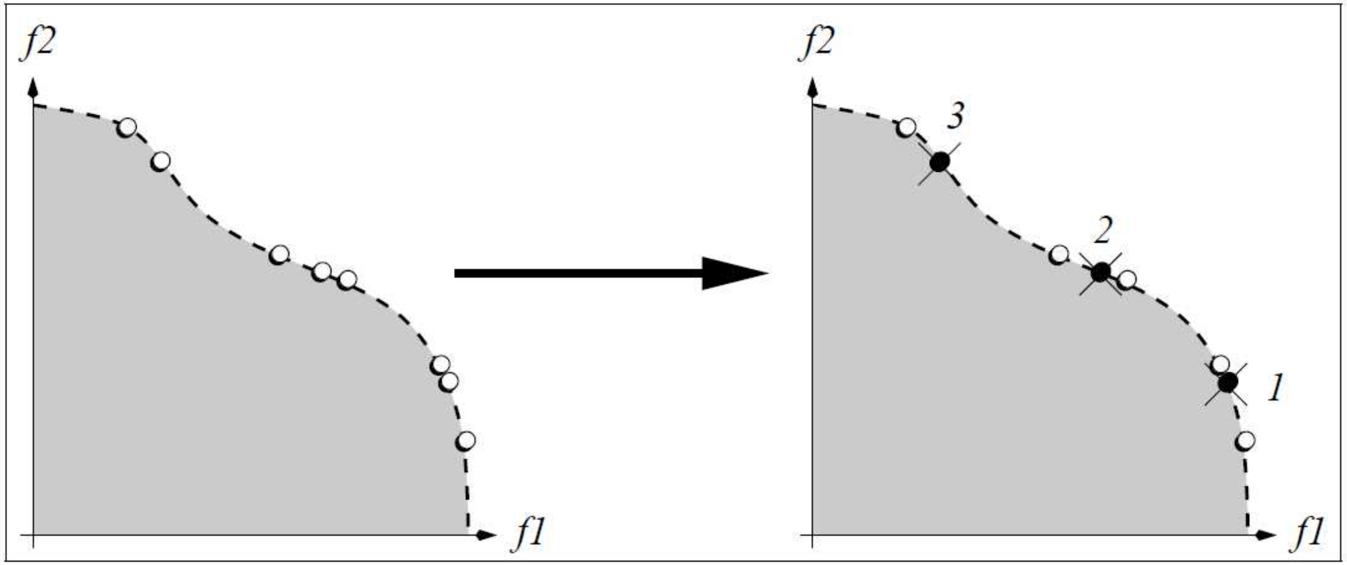
Final fitness is given as

$$F(i) = R(i) + D(i).$$

SPEA2: Environmental Selection

After copying all nondominated individuals from archive and population to the archive of the next generation,

- if the archive is too small (i.e. $|\overline{P}_{t+1} < \overline{N}|$), the best $\overline{N} - |\overline{P}_{t+1}|$ dominated solutions (w.r.t. fitness) in the previous archive and population are copied to the new archive;
- if the archive is too large (i.e. $|\overline{P}_{t+1} > \overline{N}|$), individuals from $\overline{P}_{t+1}$ are iteratively removed until $|\overline{P}_{t+1}| = \overline{N}$.
  At each iteration, the individual which has the minimum distance to another individual is chosen (a tie is broken by considering the second smallest distances and so forth).



Multi-objective Opt.

Multi-objective EAs
- MOEAs: Why?
- MOEAs
- NSGA
- Fitness Sharing
- FS: Example
- NSGA: Fitness
- NSGA: Conclusions
- NSGA-II
- NSGA-II: Diversity
- NSGA-II Steps
- NSGA vs NSGA-II
- Simulation results
- Constraints
- NSGA-II Sym. Reg.
- SPEA2
- SPEA2 Steps
- SPEA2 Fitness
- SPEA2 Diversity
- SPEA2 Replacement
- SPEA2: Conclusions

Performance Measures

Summary

SPEA2: Conclusions

Multi-objective Opt.

Multi-objective EAs
- MOEAs: Why?
- MOEAs
- NSGA
- Fitness Sharing
- FS: Example
- NSGA: Fitness
- NSGA: Conclusions
- NSGA-II
- NSGA-II: Diversity
- NSGA-II Steps
- NSGA vs NSGA-II
- Simulation results
- Constraints
- NSGA-II Sym. Reg.
- SPEA2
- SPEA2 Steps
- SPEA2 Fitness
- SPEA2 Diversity
- SPEA2 Replacement
- SPEA2: Conclusions

Performance Measures

Summary

SPEA2

- uses the concept of **Pareto dominance** in order to assign scalar fitness values to individuals;
- uses a **fine-grained fitness** assignment strategy which **incorporates density information** in order to distinguish between solutions that are indifferent, i.e., do not dominate each other;
- uses environmental selection in order to keep the optimal diversity in the archive;
- seems to have advantages over NSGA-II in higher dimensional objective spaces.

Measuring MO Performance

MOEA Performance Measures

The result of a MOEA run is not a single scalar value, but a collection of vectors forming a non-dominated set.

- Comparing two MOEA algorithms requires comparing the non-dominated sets they produce.
- However, there is no straightforward way to compare different non-dominated sets.

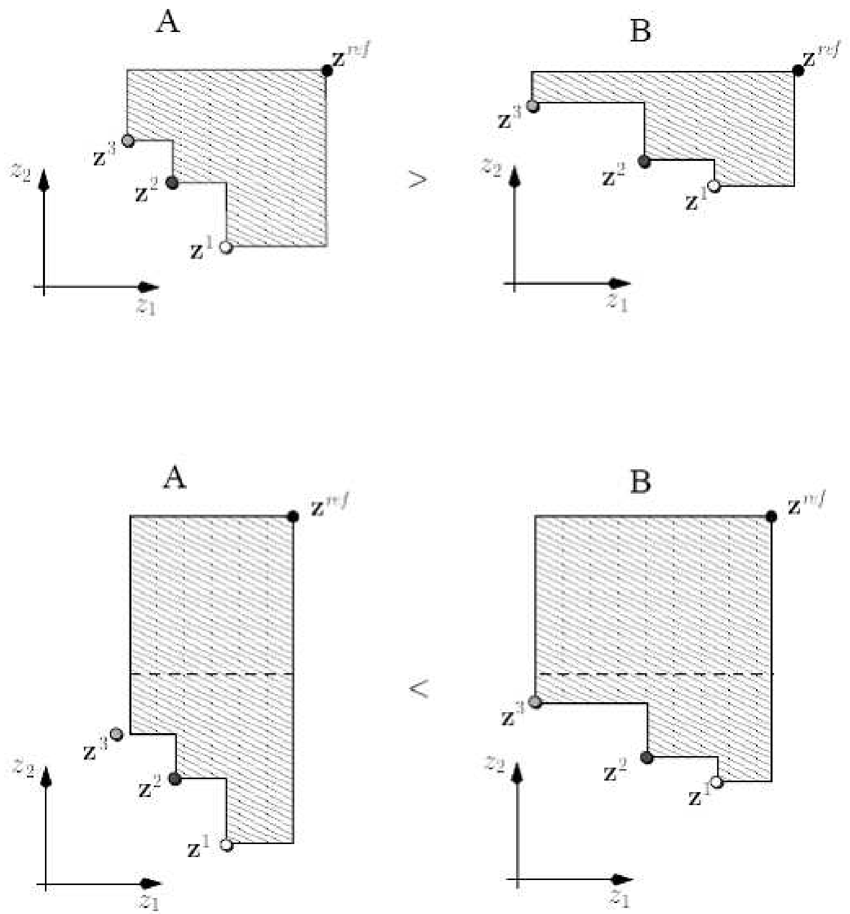
Three goals that can be identified and measured:

1. The distance of the resulting non-dominated front to the Pareto front should be minimized.
2. A good (in most cases uniform) distribution of the solutions found is desirable.
3. The extent of the obtained non-dominated front should be maximized, i.e., for each objective, a wide range of values should be present.

S Metric

Size of the space covered $S(X)$: it calculates the *hypervolume* of the multi-dimensional region enclosed by a set A and a *reference point* Z^{ref}. The hypervolume expresses the size of the region that is dominated by A.

So, the bigger the value of this measure the better the quality of A is, and vice versa.



©Knowles J. and Corne D.: On Metrics for Comparing Non-Dominated Sets.

Multi-objective Opt.
Multi-objective EAs
Performance Measures
• MOEA Performance
• S Metric
• C Metric
Summary

S Metric (cont.)

Pros:

- Given two non-dominated sets, A and B, if each point in B is dominated by a point in A then A will always be evaluated as being better than B.
- Independence: the hypervolume calculated for the given set is not dependent on any other, or any reference set.
- Differentiates between different degrees of complete outperformance of two sets.
- Intuitive meaning/interpretation.

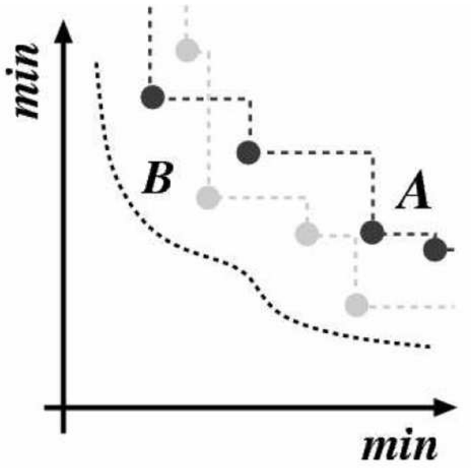
Cons:

- Requires defining some upper boundary of the region.
  This choice does affect the ordering of non-dominated sets.
- It has a large computational overhead, $O(n^{k+1})$, where n is the number of nondominated solutions and k is the number of objectives, rendering it unusable for many objectives or large sets.
- It multiplies apples by oranges, i.e., different objectives together.

C Metric

Coverage of two sets $C(X,Y)$: given two sets of non-dominated solutions X and Y found by the compared algorithms, the measure $C(X,Y)$ returns a ratio of a number of solutions of Y that are dominated by or equal to any solution of X to the whole set Y.

- It returns values from the interval [0, 1].
- The value $C(X,Y) = 1$ means that all solutions in Y are covered by solutions of the set X. And vice versa, the value $C(X,Y) = 0$ means that none of the solutions in Y are covered by the set X.
- Always both orderings have to be considered, since $C(X,Y)$ is not necessarily equal to $1 - C(Y,X)$.



$C(A,B) = 0.25$, $C(B,A) = 0.75$

Properties:

- It has low computational overhead.
- If two sets are of different cardinality and/or the distributions of the sets are non-uniform, then it gives unreliable results.

C Metric (cont.)

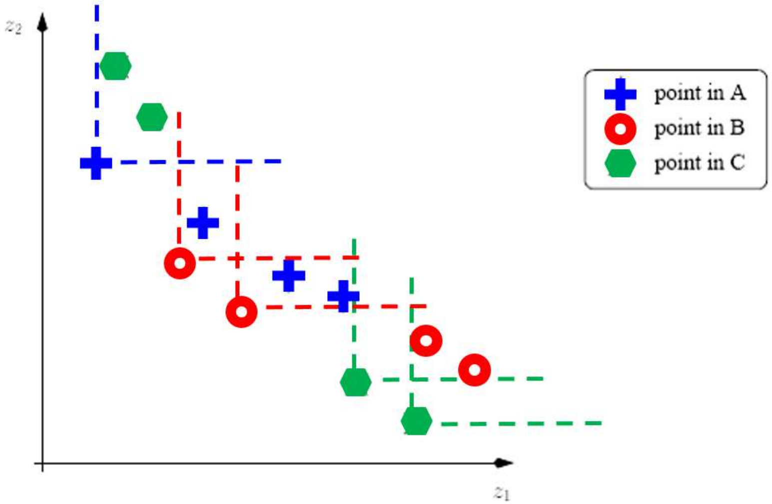
Properties:

- Any pair of C metric scores for a pair of sets A and B in which neither $C(A,B) = 1$ nor $C(B,A) = 1$, indicates that the two sets are incomparable according to the weak outperformance relation.
- It is cycleinducing – if three sets are compared using C, they may not be ordered.

Example:

- $C(A,B) = 0, C(B,A) = 3/4$
- $C(B,C) = 0, C(C,B) = 1/2$
- $C(A,C) = 1/2, C(C,A) = 0$

B considered better than A, A better than C, but C better than B.



©Knowles J. and Corne D.: On Metrics for Comparing Non-Dominated Sets.

Summary

Learning outcomes

After this lecture, a student shall be able to

- define a multi-objective optimization problem and describe the relationship between decision and objective spaces;
- define the dominance principle and the Pareto-optimal solutions;
- identify non-dominated solutions in a set of solutions;
- list and describe two goals of multi-objective optimization;
- describe some non-evolutionary approaches to multi-objective optimizatin and explain their deficiencies;
- implement evolutionary multi-objective algorithms and explain their differences from ordinary EA;
- explain algorithms NSGA, NSGA-II, SPEA2 and their differences;
- implement constraint handling in NSGA-II;
- define performance measures used in multi-objective optimizations (S metric and C metric);

Reading

- Kalyanmoy Deb:
  Multi-objective optimization using evolutionary algorithms.
  Wiley, 2001.
- Kalyanmoy Deb et al.:
  A Fast and Elitist Multiobjective Genetic Algorithm: NSGA-II.
  IEEE Transactions on Evolutionary Computation, vol. 6, pp. 182–197, 2000.
- Eckart Zitzler et al.:
  SPEA2: Improving the Strength Pareto Evolutionary Algorithm.
  ETH Zurich, 2001.
- Eckart Zitzler:
  Evolutionary Algorithms for Multiobjective Optimization: Methods and Applications.
  ETH Zurich, 1999.
- Joshua Knowles and David Corne:
  On Metrics for Comparing Non-Dominated Sets. IEEE Congress on Evolutionary Computation, 2002.