Question 1. (3 points)

Give a real-life meaning to binary random variables A, B, C, for which the following Bayes graph is appropriate

1. $\textcircled{A} \to \textcircled{B} \to \textcircled{C}$
2. $\textcircled{A} \to \textcircled{B} \leftarrow \textcircled{C}$
3. $\textcircled{A} \leftarrow \textcircled{B} \to \textcircled{C}$

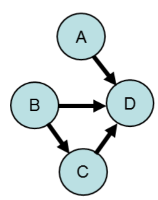
For each case, decide if the graph implies $A \perp\!\!\!\perp_P C \mid B$.

Answer:

1. A : thorough studying, B : good understanding, C : successful exam. YES
2. A : flu, B : fever, C : pneumonia. NO
3. A : high cancer incidence, B : nuclear blast, C : compromised environment. YES

Question 2. (2 points)

How many parameters are needed in the Bayesian network below to fully specify a joint distribution on the random variables in vertices, which are binary?

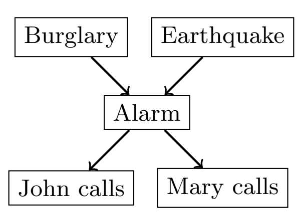


Answer:

Summing parameters in vertices in alphabetica order: $2^0 + 2^0 + 2^1 + 2^3 = 12$

Question 3. (5 points)

Given the Bayesian network below (conditional probability tables not shown),



1. (1 point) list all variables that 'John calls' is independent of, given that 'Alarm' is observed.
   Answer:
   Burglary, Earthquake, Mary calls

2. (4 points) express the probability that neither Mary nor John calls given that both burglary and earthquake happens, using only those (conditional) probabilities which are encoded in the conditional probability tables appropriate for this network. (Produce a formula referring to the random events by symbols such as A and their outcomes by symbols such as $b, \neg m$, etc.).

Answer:

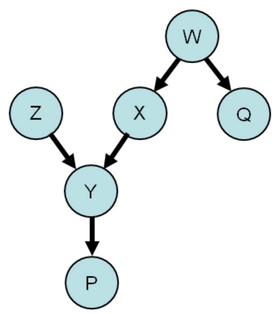
$$\frac{P(\neg m, \neg j, b, e)}{P(b, e)} =$$

$$\frac{\sum_A P(b)P(e)P(A|b,e)P(\neg j|A)P(\neg m|A)}{P(b)P(e)} =$$

$$\sum_A P(A|b,e)P(\neg j|A)P(\neg m|A).$$

Question 4. (5 points)

Let $X_1, X_2, \ldots \perp\!\!\!\perp_P Y_1, Y_2, \ldots \mid \mathcal{E}$ denote that $\forall i, j \in \{1, 2, \ldots\} : X_i \perp\!\!\!\perp_P Y_j \mid \mathcal{E}$. The empty set is denoted as $\emptyset$. Decide (true/false) for each of the statements below whether it is implied by the Bayes graph for P:



1. $Q \perp\!\!\!\perp_P X, Y, Z, P \mid W$,

   Answer:

   True, Q ← W → X, diverging connection, W observed

2. $Z, Y, P \perp\!\!\!\perp_P W, Q \mid \emptyset$,

   Answer:

   False, W → X → Y linear, X not observed

3. $Z \perp\!\!\!\perp_P X, W, Q \mid \emptyset$,

   Answer:

   True Z → Y ← X converging connection, neither Y nor P observed

4. $Z \perp\!\!\!\perp_P X, W, Q \mid P$,

   Answer:

   False, Z → Y ← X converging connection, descendant P observed

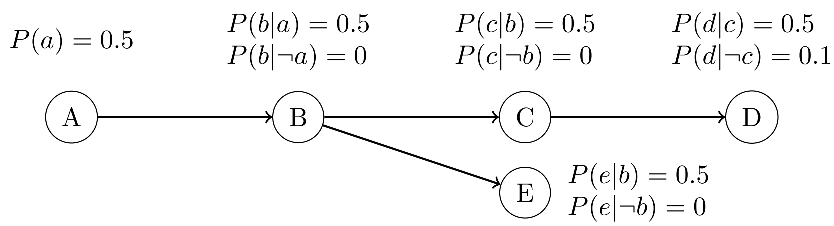
5. $Z, Y, P \perp\!\!\!\perp_P W, Q \mid X$,

   Answer:

   True, W → X → Y linear connection, X observed

Question 5. (15 points)

Consider the Bayes Network



1. Calculate $P(a|d)$ and $P(\neg a|d)$ by the factor method, i.e., using factor multiplication and variable elimination.
2. Determine $\arg\max_{A,B} P(A, B|d)$ by the factor method.

Answer:

1. E can be ignored as it is a terminal vertex which is not part of the query or evidence.

$$P(A|d) = \alpha \sum_{B,C} P(A)P(B|A)P(C|B)P(d|C)$$

Push sums to the right

$$= \alpha P(A) \sum_{B} P(B|A) \sum_{C} P(C|B)P(d|C)$$

Reformulate with factors

$$= \alpha \begin{bmatrix} 0.5 & \neg a \\ 0.5 & a \end{bmatrix} \times \sum_{B} \begin{bmatrix} 1 & \neg a, \neg b \\ 0 & \neg a, b \\ 0.5 & a, \neg b \\ 0.5 & a, b \end{bmatrix} \times \sum_{C} \begin{bmatrix} 1 & \neg b, \neg c \\ 0 & \neg b, c \\ 0.5 & b, \neg c \\ 0.5 & b, c \end{bmatrix} \times \begin{bmatrix} 0.1 & \neg c \\ 0.5 & c \end{bmatrix}$$

Multiply last two factors

$$= \alpha \begin{bmatrix} 0.5 & \neg a \\ 0.5 & a \end{bmatrix} \times \sum_{B} \begin{bmatrix} 1 & \neg a, \neg b \\ 0 & \neg a, b \\ 0.5 & a, \neg b \\ 0.5 & a, b \end{bmatrix} \times \sum_{C} \begin{bmatrix} 0.1 & \neg b, \neg c \\ 0 & \neg b, c \\ 0.05 & b, \neg c \\ 0.25 & b, c \end{bmatrix}$$

Eliminate C:

$$= \alpha \begin{bmatrix} 0.5 & \neg a \\ 0.5 & a \end{bmatrix} \times \sum_{B} \begin{bmatrix} 1 & \neg a, \neg b \\ 0 & \neg a, b \\ 0.5 & a, \neg b \\ 0.5 & a, b \end{bmatrix} \times \begin{bmatrix} 0.1 & \neg b \\ 0.3 & b \end{bmatrix}$$

Multiply last two factors

$$= \alpha \begin{bmatrix} 0.5 & \neg a \\ 0.5 & a \end{bmatrix} \times \sum_{B} \begin{bmatrix} 0.1 & \neg a, \neg b \\ 0 & \neg a, b \\ 0.05 & a, \neg b \\ 0.15 & a, b \end{bmatrix}$$

Eliminate B:

$$= \alpha \begin{bmatrix} 0.5 & \neg a \\ 0.5 & a \end{bmatrix} \times \begin{bmatrix} 0.1 & \neg a \\ 0.2 & a \end{bmatrix}$$

$$= \alpha \underbrace{\begin{bmatrix} 0.05 & \neg a \\ 0.1 & a \end{bmatrix}}_{=f(A)}$$

$P(a) + P(\neg a) = \alpha f(a) + \alpha f(\neg a) = 1$. Since $\alpha f(a) = 2\alpha f(\neg a)$, we have $P(a) = 2/3$ and $P(\neg a) = 1/3$.

2. Reuse the computation above up until the elimination of B (since B will be maximized-out instead):

$$\max_{A,B} P(A,B|d) = \alpha \max_A \begin{bmatrix} 0.5 & \neg a \\ 0.5 & a \end{bmatrix} \times \max_B \underbrace{\begin{bmatrix} 0.1 & \neg a, \neg b \\ 0 & \neg a, b \\ 0.05 & a, \neg b \\ 0.15 & a, b \end{bmatrix}}_{=f_1(A,B)}$$

$$\text{Maximize-out } B: \quad = \alpha \max_A \begin{bmatrix} 0.5 & \neg a \\ 0.5 & a \end{bmatrix} \times \begin{bmatrix} 0.1 & \neg a \\ 0.15 & a \end{bmatrix}$$

$$= \alpha \max_A \underbrace{\begin{bmatrix} 0.05 & \neg a \\ 0.075 & a \end{bmatrix}}_{f_2(A)}$$

$\arg\max_A f_2(A) = a$

$\arg\max_B f_1(a,B) = b$

So $\arg\max_{A,B} P(A,B|d) = (a,b)$