

3D Computer Vision

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Open Informatics Master's Course

Computing with a Single Camera

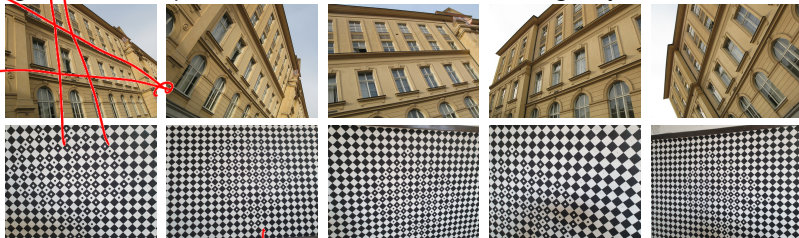
- 3.1 Calibration: Internal Camera Parameters from Vanishing Points and Lines
- 3.2 Camera Resection: Projection Matrix from 6 Known Points
- 3.3 Exterior Orientation: Camera Rotation and Translation from 3 Known Points
- 3.4 Relative Orientation Problem: Rotation and Translation between Two Point Sets

covered by

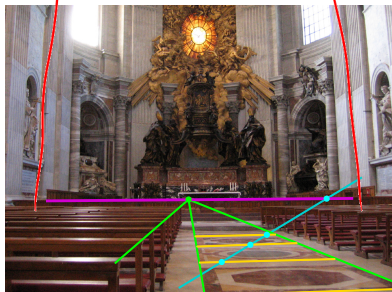
- [1] [H&Z] Secs: 8.6, 7.1, 22.1
- [2] Fischler, M.A. and Bolles, R.C . Random Sample Consensus: A Paradigm for Model Fitting with Applications to Image Analysis and Automated Cartography. *Communications of the ACM* 24(6):381–395, 1981
- [3] [Golub & van Loan 2013, Sec. 2.5]

Obtaining Vanishing Points and Lines

- orthogonal direction pairs can be collected from more images by camera rotation

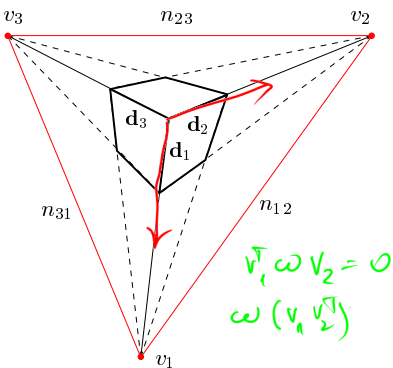


- vanishing line can be obtained from vanishing points and/or regularities (→49)



► Camera Calibration from Vanishing Points and Lines

Problem: Given finite vanishing points and/or vanishing lines, compute \mathbf{K}



$$\mathbf{d}_i = \lambda_i \mathbf{Q}^{-1} \mathbf{v}_i, \quad \frac{1}{\lambda_i} \mathbf{Q} \mathbf{d}_i = \mathbf{v}_i, \quad i = 1, 2, 3 \quad \rightarrow 43 \quad (2)$$

$$\mathbf{p}_{ij} = \mu_{ij} \mathbf{Q}^T \mathbf{n}_{ij}, \quad i, j = 1, 2, 3, \quad i \neq j \quad \rightarrow 39$$

- simple method: solve (2) after eliminating λ_i, μ_{ij} .

Special Configurations $(\mathbf{Q} \mathbf{Q}^T)^{-1} = (\mathbf{K} \mathbf{B} \mathbf{B}^T \mathbf{K}^T)^{-1}$

1. orthogonal rays $\mathbf{d}_1 \perp \mathbf{d}_2$ in space then

$$0 = \mathbf{d}_1^T \mathbf{d}_2 = \mathbf{v}_1^T \mathbf{Q}^{-T} \mathbf{Q}^{-1} \mathbf{v}_2 = \mathbf{v}_1^T \underbrace{(\mathbf{K} \mathbf{K}^T)^{-1}}_{\omega \text{ (IAC)}} \mathbf{v}_2$$

2. orthogonal planes $\mathbf{p}_{ij} \perp \mathbf{p}_{ik}$ in space

$$0 = \mathbf{p}_{ij}^T \mathbf{p}_{ik} = \mathbf{n}_{ij}^T \mathbf{Q} \mathbf{Q}^T \mathbf{n}_{ik} = \mathbf{n}_{ij}^T \omega^{-1} \mathbf{n}_{ik}$$

3. orthogonal ray and plane $\mathbf{d}_k \parallel \mathbf{p}_{ij}, k \neq i, j$ normal parallel to optical ray

$$\mathbf{p}_{ij} \simeq \mathbf{d}_k \Rightarrow \mathbf{Q}^T \mathbf{n}_{ij} = \frac{\lambda_i}{\mu_{ij}} \mathbf{Q}^{-1} \mathbf{v}_k \Rightarrow \mathbf{n}_{ij} = \kappa \mathbf{Q}^{-T} \mathbf{Q}^{-1} \mathbf{v}_k = \kappa \omega \mathbf{v}_k, \quad \kappa \neq 0$$

$k \neq \{i, j\}$

- n_{ij} may be constructed from non-orthogonal v_i and v_j , e.g. using the cross-ratio

- ω is a symmetric, positive definite 3×3 matrix

IAC = Image of Absolute Conic

► cont'd

configuration	equation	# constraints
(3) orthogonal v.p.	$\underline{\mathbf{v}}_i^\top \boldsymbol{\omega} \underline{\mathbf{v}}_j = 0$	1
(4) orthogonal v.l.	$\underline{\mathbf{n}}_{ij}^\top \boldsymbol{\omega}^{-1} \underline{\mathbf{n}}_{ik} = 0$	1
(5) v.p. orthogonal to v.l.	$\underline{\mathbf{n}}_{ij} = \varkappa \boldsymbol{\omega} \underline{\mathbf{v}}_k$	2
(6) orthogonal image raster $\theta = \pi/2$	$\omega_{12} = \omega_{21} = 0$	1
(7) unit aspect $a = 1$ when $\theta = \pi/2$	$\omega_{11} - \omega_{22} = 0$	1
(8) known principal point $u_0 = v_0 = 0$	$\omega_{13} = \omega_{31} = \omega_{23} = \omega_{32} = 0$	2

$\boldsymbol{\omega} \in \mathbb{R}^5$

- these are homogeneous linear equations for the 5 parameters in $\boldsymbol{\omega}$ in the form $\mathbf{D}\boldsymbol{\omega} = \mathbf{0}$
 \varkappa can be eliminated from (5)
- we need at least 5 constraints for full $\boldsymbol{\omega}$ symmetric 3×3
- we get \mathbf{K} from $\boldsymbol{\omega}^{-1} = \mathbf{K}\mathbf{K}^\top$ by Choleski decomposition
the decomposition returns a positive definite upper triangular matrix
one avoids solving an explicit set of quadratic equations for the parameters in \mathbf{K}

Examples

Assuming orthogonal raster, unit aspect (ORUA): $\theta = \pi/2$, $a = 1$

$$\omega \simeq \begin{bmatrix} 1 & 0 & -u_0 \\ 0 & 1 & -v_0 \\ -u_0 & -v_0 & f^2 + u_0^2 + v_0^2 \end{bmatrix}$$

Ex 1:

Assuming ORUA and known $m_0 = (u_0, v_0)$, two finite orthogonal vanishing points give f

$$\underline{\mathbf{v}_1^\top \omega \mathbf{v}_2} = 0 \quad \Rightarrow \quad f^2 = |(\mathbf{v}_1 - \mathbf{m}_0)^\top (\mathbf{v}_2 - \mathbf{m}_0)|$$

in this formula, \mathbf{v}_i , \mathbf{m}_0 are Cartesian (not homogeneous)!

Ex 2:

Non-orthogonal vanishing points \mathbf{v}_i , \mathbf{v}_j , known angle ϕ : $\cos \phi = \frac{\mathbf{v}_i^\top \omega \mathbf{v}_j}{\sqrt{\mathbf{v}_i^\top \omega \mathbf{v}_i} \sqrt{\mathbf{v}_j^\top \omega \mathbf{v}_j}}$

- leads to polynomial equations
- e.g. ORUA and $u_0 = v_0 = 0$ gives

$$(f^2 + \mathbf{v}_i^\top \mathbf{v}_j)^2 = (f^2 + \|\mathbf{v}_i\|^2) \cdot (f^2 + \|\mathbf{v}_j\|^2) \cdot \cos^2 \phi$$

► Camera Orientation from Two Finite Vanishing Points

Problem: Given \mathbf{K} and two vanishing points corresponding to two known orthogonal directions $\mathbf{d}_1, \mathbf{d}_2$, compute camera orientation \mathbf{R} with respect to the plane.

- 3D coordinate system choice, e.g.:

$$\mathbf{d}_1 = (1, 0, 0), \quad \mathbf{d}_2 = (0, 1, 0)$$

- we know that

$$\mathbf{d}_i \simeq \mathbf{Q}^{-1} \underline{\mathbf{v}}_i = (\mathbf{K}\mathbf{R})^{-1} \underline{\mathbf{v}}_i = \mathbf{R}^{-1} \underbrace{\mathbf{K}^{-1} \underline{\mathbf{v}}_i}_{\underline{\mathbf{w}}_i}$$

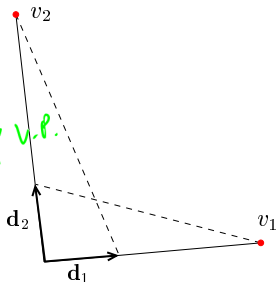
$$\mathbf{R} \mathbf{d}_i \simeq \underline{\mathbf{w}}_i$$

- knowing $\mathbf{d}_{1,2}$ we conclude that $\underline{\mathbf{w}}_i / \|\underline{\mathbf{w}}_i\|$ is the i -th column \mathbf{r}_i of \mathbf{R}
- the third column is orthogonal:

$$\mathbf{r}_3 \simeq \mathbf{r}_1 \times \mathbf{r}_2$$

$$\mathbf{R} = \begin{bmatrix} \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} & \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} & \frac{\mathbf{w}_1 \times \mathbf{w}_2}{\|\mathbf{w}_1 \times \mathbf{w}_2\|} \end{bmatrix}$$

- in general we have to care about the signs $\pm \underline{\mathbf{w}}_i$ (such that $\det \mathbf{R} = 1$)



some suitable scenes



Application: Planar Rectification

Principle: Rotate camera (image plane) parallel to the plane of interest.



$$\underline{\mathbf{m}} \simeq \mathbf{K}\mathbf{R} \left[\mathbf{I} \quad -\mathbf{C} \right] \underline{\mathbf{X}}$$

$\mathbf{K}(\mathbf{K}\mathbf{R})^{-1}$ (handwritten green annotations)

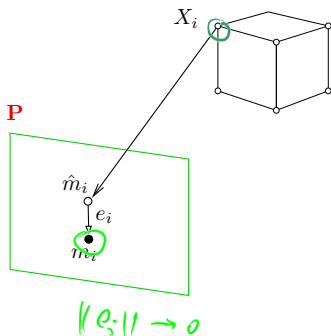
$$\underline{\mathbf{m}}' \simeq \mathbf{K}' \left[\mathbf{I} \quad -\mathbf{C}' \right] \underline{\mathbf{X}}$$
$$\underline{\mathbf{m}}' \simeq \mathbf{K}(\mathbf{K}\mathbf{R})^{-1} \underline{\mathbf{m}} = \mathbf{K}\mathbf{R}^T \mathbf{K}^{-1} \underline{\mathbf{m}} = \mathbf{H} \underline{\mathbf{m}}$$

(handwritten green annotations)

- \mathbf{H} is the rectifying homography
- both \mathbf{K} and \mathbf{R} can be calibrated from two finite vanishing points [assuming ORUA](#) →57
- not possible when one of them is (or both are) infinite
- without ORUA we would need 4 additional views to calibrate \mathbf{K} as on →54

► Camera Resection

Camera calibration and orientation from a known set of $k \geq 6$ reference points and their images $\{(X_i, m_i)\}_{i=1}^6$.

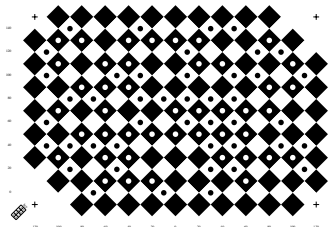


- X_i are considered exact
- m_i is a measurement subject to detection error

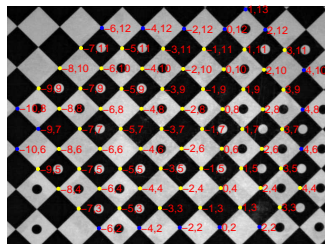
$$\mathbf{m}_i = \hat{\mathbf{m}}_i + \mathbf{e}_i \quad \text{Cartesian}$$

- where $\lambda_i \hat{\mathbf{m}}_i = \mathbf{P}\underline{\mathbf{X}}_i$

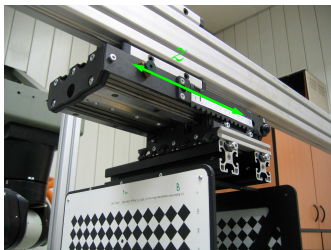
Resection Targets



calibration chart



automatic calibration point detection



resection target with translation stage

- target translated at least once
- by a calibrated (known) translation
- X_i point locations looked up in a table based on their code

► The Minimal Problem for Camera Resection

Problem: Given $k = 6$ corresponding pairs $\{(X_i, m_i)\}_{i=1}^k$, find \mathbf{P}

$$\lambda_i \underline{m}_i = \mathbf{P} \underline{X}_i, \quad \mathbf{P} = \begin{bmatrix} \mathbf{q}_1^\top & q_{14} \\ \mathbf{q}_2^\top & q_{24} \\ \mathbf{q}_3^\top & q_{34} \end{bmatrix} \quad \underline{X}_i = (x_i, y_i, z_i, 1), \quad i = 1, 2, \dots, k, \quad k = 6$$

$$\underline{m}_i = (u_i, v_i, 1), \quad \lambda_i \in \mathbb{R}, \quad \lambda_i \neq 0, \quad |\lambda_i| < \infty$$

easily modifiable for infinite points X_i but be aware of $\rightarrow 64$

expanded: $\lambda_i u_i = \mathbf{q}_1^\top \underline{X}_i + q_{14}, \quad \lambda_i v_i = \mathbf{q}_2^\top \underline{X}_i + q_{24}, \quad \lambda_i = \mathbf{q}_3^\top \underline{X}_i + q_{34}$

after elimination of λ_i : $(\mathbf{q}_3^\top \underline{X}_i + q_{34})u_i = \mathbf{q}_1^\top \underline{X}_i + q_{14}, \quad (\mathbf{q}_3^\top \underline{X}_i + q_{34})v_i = \mathbf{q}_2^\top \underline{X}_i + q_{24}$

Then

$$\mathbf{A} \mathbf{q} = \begin{bmatrix} \mathbf{X}_1^\top & 1 & \mathbf{0}^\top & 0 & -u_1 \mathbf{X}_1^\top & -u_1 \\ \mathbf{0}^\top & 0 & \mathbf{X}_1^\top & 1 & -v_1 \mathbf{X}_1^\top & -v_1 \\ \vdots & & & & & \\ \mathbf{X}_k^\top & 1 & \mathbf{0}^\top & 0 & -u_k \mathbf{X}_k^\top & -u_k \\ \mathbf{0}^\top & 0 & \mathbf{X}_k^\top & 1 & -v_k \mathbf{X}_k^\top & -v_k \end{bmatrix} \cdot \begin{bmatrix} \mathbf{q}_1 \\ q_{14} \\ \mathbf{q}_2 \\ q_{24} \\ \mathbf{q}_3 \\ q_{34} \end{bmatrix} = \mathbf{0} \quad (9)$$

12 × 12

- we need 11 independent parameters for \mathbf{P}
- $\mathbf{A} \in \mathbb{R}^{2k, 12}$, $\mathbf{q} \in \mathbb{R}^{12}$
- 6 points in a general position give $\text{rank } \mathbf{A} = 12$ and there is no (non-trivial) null space
- drop one row to get rank-11 matrix, then the basis vector of the null space of \mathbf{A} gives \mathbf{q}

► The Jack-Knife Solution for $k = 6$

- given the 6 correspondences, we have 12 equations for the 11 parameters
- can we use all the information present in the 6 points?

Jack-knife estimation

1. $n := 0$
2. for $i = 1, 2, \dots, 2k$ do
 - a) delete i -th row from \mathbf{A} , this gives \mathbf{A}_i
 - b) if $\dim \text{null } \mathbf{A}_i > 1$ continue with the next i
 - c) $n := n + 1$
 - d) compute the right null-space \mathbf{q}_i of \mathbf{A}_i
 - e) $\hat{\mathbf{q}}_i := \mathbf{q}_i$ normalized to $q_{34} = 1$ and dimension-reduced
3. from all n vectors $\hat{\mathbf{q}}_i$ collected in Step 1d compute

$$\mathbf{q} = \frac{1}{n} \sum_{i=1}^n \hat{\mathbf{q}}_i, \quad \text{var}[\mathbf{q}] = \frac{n-1}{n} \text{diag} \sum_{i=1}^n (\hat{\mathbf{q}}_i - \mathbf{q})(\hat{\mathbf{q}}_i - \mathbf{q})^\top$$

regular for $n \geq 11$
variance of the sample mean

- have a solution + an error estimate, per individual elements of \mathbf{P} (except P_{34})
- at least 5 points must be in a general position ($\rightarrow 64$)
- large error indicates near degeneracy
- computation not efficient with $k > 6$ points, needs $\binom{2k}{11}$ draws, e.g. $k = 7 \Rightarrow 364$ draws
- better error estimation method: decompose \mathbf{P}_i to $\mathbf{K}_i, \mathbf{R}_i, \mathbf{t}_i$ ($\rightarrow 33$), represent \mathbf{R}_i with 3 parameters (e.g. Euler angles, or in Cayley representation $\rightarrow 141$) and compute the errors for the parameters
- even better: use the SE(3) Lie group for $(\mathbf{R}_i, \mathbf{t}_i)$ and average its Lie-algebra representations



e.g. by 'economy-size' SVD
assuming finite cam. with $P_{3,4} = 1$

► Degenerate (Critical) Configurations for Camera Resection

Let $\mathcal{X} = \{X_i; i = 1, \dots\}$ be a set of points and $\mathbf{P}_1 \neq \mathbf{P}_j$ be two regular (rank-3) cameras. Then two configurations $(\mathbf{P}_1, \mathcal{X})$ and $(\mathbf{P}_j, \mathcal{X})$ are image-equivalent if

$$\mathbf{P}_1 \underline{X}_i \simeq \mathbf{P}_j \underline{X}_i \quad \text{for all } X_i \in \mathcal{X}$$

$$\mathbf{P}_1, \mathbf{T}_j, j+1$$

there is a non-trivial set of other cameras that see the same image

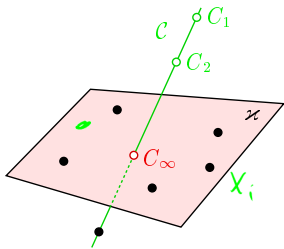
Results

- importantly: If all calibration points $X_i \in \mathcal{X}$ lie on a plane π then camera resection is non-unique and all image-equivalent camera centers lie on a spatial line \mathcal{C} with the $C_\infty = \pi \cap \mathcal{C}$ excluded

this also means we cannot resect if all X_i are infinite

- and more: by adding points $X_i \in \mathcal{X}$ to \mathcal{C} we gain nothing
- there are additional image-equivalent configurations, see next

proof sketch in [H&Z, Sec. 22.1.2]

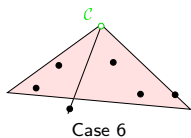
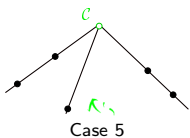


Case 4

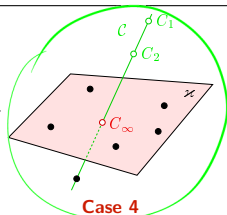
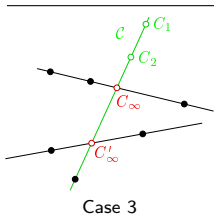
Note that if \mathbf{Q}, \mathbf{T} are suitable homographies then $\mathbf{P}_1 \simeq \mathbf{Q}\mathbf{P}_0\mathbf{T}$, where \mathbf{P}_0 is canonical and the analysis can be made with $\hat{\mathbf{P}}_j \simeq \mathbf{Q}^{-1}\mathbf{P}_j$

$$\mathbf{P}_0 \underbrace{\mathbf{T}\underline{X}_i}_{\underline{Y}_i} \simeq \hat{\mathbf{P}}_j \underbrace{\mathbf{T}\underline{X}_i}_{\underline{Y}_i} \quad \text{for all } Y_i \in \mathcal{Y}$$

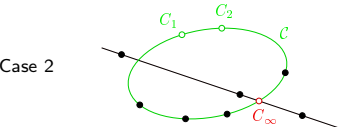
cont'd (all cases)



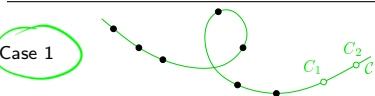
- cameras C_1, C_2 co-located at point C
- points on three optical rays or one optical ray and one optical plane
- Case 5: camera sees 3 isolated point images
- Case 6: cam. sees a line of points and an isolated point



- cameras lie on a line $C \setminus \{C_\infty, C'_\infty\}$
- points lie on C and
 1. on two lines meeting C at C_∞, C'_∞
 2. or on a plane meeting C at C_∞
- Case 3: camera sees 2 lines of points
- Case 4: **dangerous!**



- cameras lie on a planar conic $C \setminus \{C_\infty\}$
not necessarily an ellipse
- points lie on C and an additional line meeting the conic at C_∞
- Case 2: camera sees 2 lines of points



- cameras and points all lie on a twisted cubic C
- Case 1: camera sees points on a conic
dangerous but unlikely

► Three-Point Exterior Orientation Problem (P3P)

Calibrated camera rotation and translation from Perspective images of 3 reference Points.

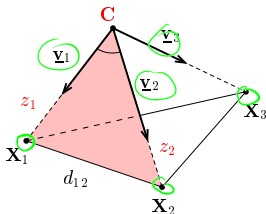
Problem: Given \mathbf{K} and three corresponding pairs $\{(m_i, X_i)\}_{i=1}^3$, find \mathbf{R} , \mathbf{C} by solving

$$\lambda_i \underline{\mathbf{m}}_i = \mathbf{K} \mathbf{R} (\mathbf{X}_i - \mathbf{C}), \quad i = 1, 2, 3 \quad \mathbf{X}_i \text{ Cartesian}$$

1. Transform $\underline{\mathbf{v}}_i \stackrel{\text{def}}{=} \mathbf{K}^{-1} \underline{\mathbf{m}}_i$. Then

$$\lambda_i \underline{\mathbf{v}}_i = \mathbf{R} (\mathbf{X}_i - \mathbf{C}). \quad (10)$$

2. If there was no rotation in (10), the situation would look like this



3. and we could shoot 3 lines from the given points \mathbf{X}_i in given directions $\underline{\mathbf{v}}_i$ to get \mathbf{C}
4. given \mathbf{C} we solve (10) for λ_i , \mathbf{R}

►P3P cont'd

If there is rotation \mathbf{R}

1. Eliminate \mathbf{R} by taking rotation preserves length: $\|\mathbf{R}\mathbf{x}\| = \|\mathbf{x}\|$

$$|\lambda_i| \cdot \|\mathbf{v}_i\| = \|\mathbf{X}_i - \mathbf{C}\| \stackrel{\text{def}}{=} z_i \quad (11)$$

2. Consider only angles among \mathbf{v}_i and apply Cosine Law per triangle $(\mathbf{C}, \mathbf{X}_i, \mathbf{X}_j)$ $i, j = 1, 2, 3, i \neq j$

$$d_{ij}^2 = z_i^2 + z_j^2 - 2 z_i z_j c_{ij},$$

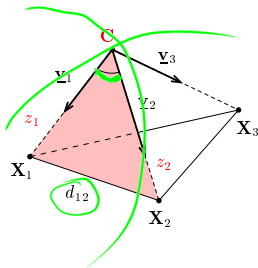
$$z_i = \|\mathbf{X}_i - \mathbf{C}\|, \quad d_{ij} = \|\mathbf{X}_j - \mathbf{X}_i\|, \quad c_{ij} = \cos(\angle \mathbf{v}_i \mathbf{v}_j)$$

4. Solve system of 3 quadratic eqs in 3 unknowns z_i

[Fischler & Bolles, 1981]

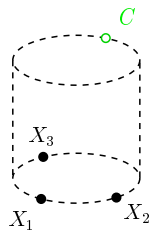
there may be no real root; there are up to 4 solutions that cannot be ignored
(verify on additional points)

5. Compute \mathbf{C} by trilateration (3-sphere intersection) from \mathbf{X}_i and z_i ; then λ_i from (11) and \mathbf{R} from (10)



Similar problems (P4P with unknown f) at <http://cmp.felk.cvut.cz/minimal/> (with code)

Degenerate (Critical) Configurations for Exterior Orientation



unstable solution

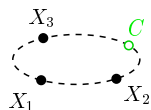
- center of projection C located on the orthogonal circular cylinder with base circumscribing the three points X_i

unstable: a small change of X_i results in a large change of C
can be detected by error propagation

degenerate

- camera C is coplanar with points (X_1, X_2, X_3) but is not on the circumscribed circle of (X_1, X_2, X_3)

camera sees points on a line



no solution

- C cocyclic with (X_1, X_2, X_3) camera sees points on a line

- additional critical configurations depend on the quadratic equations solver

[Haralick et al. IJCV 1994]

► Populating A Little ZOO of Minimal Geometric Problems in CV

problem	given	unknown	slide
camera resection	6 world–img correspondences $\{(X_i, m_i)\}_{i=1}^6$	P	62
exterior orientation	K , 3 world–img correspondences $\{(X_i, m_i)\}_{i=1}^3$	R, C	66
relative orientation	3 world–world correspondences $\{(X_i, Y_i)\}_{i=1}^3$	R, t	70

- camera resection and exterior orientation are similar problems in a sense:
 - we do resectioning when our camera is uncalibrated
 - we do orientation when our camera is calibrated
- relative orientation involves no camera (see next)
- more problems to come

Thank You



