

# 3D Computer Vision

Radim Šára    Martin Matoušek

Center for Machine Perception  
Department of Cybernetics  
Faculty of Electrical Engineering  
Czech Technical University in Prague

<https://cw.fel.cvut.cz/wiki/courses/tdv/start>

<http://cmp.felk.cvut.cz>

<mailto:sara@cmp.felk.cvut.cz>

phone ext. 7203

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Open Informatics Master's Course

## ► Local Optimization for Fundamental Matrix Estimation

### Summary so far

- Given a set  $X = \{(x_i, y_i)\}_{i=1}^k$  of  $k \gg 7$  inlier correspondences, compute a statistically efficient estimate for fundamental matrix  $\mathbf{F}$ .
  1. Find the conditioned ( $\rightarrow 92$ ) 7-point  $\mathbf{F}_0$  ( $\rightarrow 84$ ) from a suitable 7-tuple
  2. Improve the  $\mathbf{F}_0^*$  using the LM optimization ( $\rightarrow 107-108$ ) and the Sampson error ( $\rightarrow 109$ ) on all inliers, reinforce rank-2, unit-norm  $\mathbf{F}_k^*$  after each LM iteration using SVD

### Partial conceptualization

- inlier = correspondence
- outlier = non-correspondence
- binary inlier/outlier labels are hidden
- we can get a likely estimate only, with respect to a model

### We are not yet done

- if there are no wrong correspondences (mismatches, outliers), this gives a local optimum given the 7-point initial estimate
- the algorithm breaks under contamination of (inlier) correspondences by outliers
- the full problem involves finding the inliers!
- in addition, we need a mechanism for jumping out of local minima (and exploring the space of all fundamental matrices)

# ► The Full Problem of Matching and Fundamental Matrix Estimation

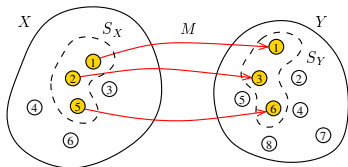
**Problem:** Given image point sets  $X = \{x_i\}_{i=1}^m$  and  $Y = \{y_j\}_{j=1}^n$  and their descriptors  $D$ , find the most probable

1. inlier keypoints  $S_X \subseteq X$ ,  $S_Y \subseteq Y$
2. one-to-one perfect matching  $M: S_X \rightarrow S_Y$
3. fundamental matrix  $F$  such that  $\text{rank } F = 2$
4. such that for each  $x_i \in S_X$  and  $y_j = M(x_i)$  it is probable that
  - a) the image descriptor  $D(x_i)$  is similar to  $D(y_j)$ , and
  - b) the total reprojection error  $E = \sum_{ij} e_{ij}^2(F)$  is small
5. inlier-outlier and outlier-outlier matches are improbable

perfect matching: 1-factor of the bipartite graph

$D(x_i) \in \mathbb{R}^d$

note a slight change in notation:  $e_{ij}$



$M:$

	Y								
X	1	2	3	4	5	6	7	8	
1	1								
2									
3									
4									
5									
6									

matching table

= 0

= 1 (matched)

$$(M^*, F^*) = \arg \max_{M, F} p(E, D, F | M) P(M) \quad (22)$$

- probabilistic model: an efficient language for problem formulation it also unifies 4.a and 4.b
- the (22) is a Bayesian probabilistic model there is a constant number of random variables!
- binary matching table  $M_{ij} \in \{0, 1\}$  of fixed size  $m \times n$ 
  - each row/column contains at most one unity
  - zero rows/columns correspond to unmatched point  $x_i/y_j$

# Deriving A Robust Matching Model by Approximate Marginalization

For algorithmic efficiency, instead of  $(M^*, \mathbf{F}^*) = \arg \max_{M, \mathbf{F}} p(E, D, \mathbf{F} | M) P(M)$  solve

$$\mathbf{F}^* = \arg \max_{\mathbf{F}} p(E, D, \mathbf{F}) \quad (23)$$

by marginalization of  $p(E, D, \mathbf{F} | M) P(M)$  over  $M$

this changes the problem!

drop the assumption that  $M$  are 1:1 matchings, assume correspondence-wise independence:

$$p(E, D, \mathbf{F} | M) P(M) = \prod_{i=1}^m \prod_{j=1}^n p_e(e_{ij}, d_{ij}, \mathbf{F} | m_{ij}) P(m_{ij})$$

- $e_{ij}$  represents (reprojection) error for match  $x_i \leftrightarrow y_j$ :  $e_{ij}(x_i, y_j, \mathbf{F})$
- $d_{ij}$  represents descriptor similarity for match  $x_i \leftrightarrow y_j$ :  $d_{ij} = \|\mathbf{d}(x_i) - \mathbf{d}(y_j)\|$

Approximate marginalization:

take all the  $2^{mn}$  terms in place of  $M$

$$\begin{aligned} p(E, D, \mathbf{F}) &\approx \sum_{m_{11} \in \{0,1\}} \sum_{m_{12}} \cdots \sum_{m_{mn}} p(E, D, \mathbf{F} | M) P(M) = \\ &= \sum_{m_{11}} \cdots \sum_{m_{mn}} \prod_{i=1}^m \prod_{j=1}^n p_e(e_{ij}, d_{ij}, \mathbf{F} | m_{ij}) P(m_{ij}) = \overset{*}{\dots} \overset{!}{=} \\ &= \prod_{i=1}^m \prod_{j=1}^n \left( \sum_{m_{ij} \in \{0,1\}} p_e(e_{ij}, d_{ij}, \mathbf{F} | m_{ij}) P(m_{ij}) \right) \end{aligned}$$

we will continue with this term

## Robust Matching Model (cont'd)

$$\begin{aligned}
 & \sum_{m_{ij} \in \{0,1\}} p_e(e_{ij}, d_{ij}, \mathbf{F} \mid m_{ij}) P(m_{ij}) = \\
 & = \underbrace{p_e(e_{ij}, d_{ij}, \mathbf{F} \mid m_{ij} = 1)}_{p_1(e_{ij}, d_{ij}, \mathbf{F})} \underbrace{P(m_{ij} = 1)}_{1-P_0} + \underbrace{p_e(e_{ij}, d_{ij}, \mathbf{F} \mid m_{ij} = 0)}_{p_0(e_{ij}, d_{ij}, \mathbf{F})} \underbrace{P(m_{ij} = 0)}_{P_0} = \\
 & = (1 - P_0) p_1(e_{ij}, d_{ij}, \mathbf{F}) + P_0 p_0(e_{ij}, d_{ij}, \mathbf{F}) \quad (24)
 \end{aligned}$$

- the  $p_0(e_{ij}, d_{ij}, \mathbf{F})$  is a penalty for 'missing a correspondence' but it should be a p.d.f. (cannot be a constant)  $(\rightarrow 114 \text{ for a simplification})$

choose  $P_0 \rightarrow 1$ ,  $p_0(\cdot) \rightarrow 0$  so that  $\frac{P_0}{1 - P_0} p_0(\cdot) \approx \text{const}$

- the  $p_1(e_{ij}, d_{ij}, \mathbf{F})$  is typically an easy-to-design term: assuming independence of reprojection error and descriptor similarity:

$$p_1(e_{ij}, d_{ij}, \mathbf{F}) = p_1(e_{ij} \mid \mathbf{F}) p_F(\mathbf{F}) p_1(d_{ij})$$

- we choose, e.g.

$$p_1(e_{ij} \mid \mathbf{F}) = \frac{1}{T_e(\sigma_1)} e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_1^2}}, \quad p_1(d_{ij}) = \frac{1}{T_d(\sigma_d, \dim \mathbf{d})} e^{-\frac{\|d(x_i) - d(y_j)\|^2}{2\sigma_d^2}} \quad (25)$$

- $\mathbf{F}$  is a random variable and  $\sigma_1$ ,  $\sigma_d$ ,  $P_0$  are parameters
- the form of  $T(\sigma_1)$  depends on error definition, it may depend on  $x_i$ ,  $y_j$  but not on  $\mathbf{F}$
- we will continue with the result from (24)

## ► Simplified Robust Energy (Error) Function

- assuming the choice of  $p_1$  as in (25), we are simplifying

$$\begin{aligned} p(E, D, \mathbf{F}) &= p(E, D \mid \mathbf{F}) p_F(\mathbf{F}) = \\ &= p_F(\mathbf{F}) \prod_{i=1}^m \prod_{j=1}^n \left[ (1 - P_0) p_1(e_{ij}, d_{ij} \mid \mathbf{F}) + P_0 p_0(e_{ij}, d_{ij} \mid \mathbf{F}) \right] \end{aligned}$$

- we choose  $\sigma_0 \gg \sigma_1$  and omit  $d_{ij}$  for simplicity; then the square-bracket term is

$$\frac{1 - P_0}{T_e(\sigma_1)} \left[ e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_1^2}} + \frac{P_0}{1 - P_0} \frac{T_e(\sigma_1)}{T_e(\sigma_0)} e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_0^2}} \right]$$

- we define the 'potential function' as:  $V(x) = -\log p(x)$ , then

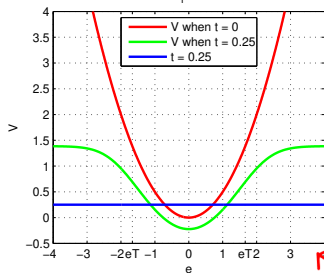
$$\begin{aligned} V(E, D \mid \mathbf{F}) &= \sum_{i=1}^m \sum_{j=1}^n \left[ \underbrace{-\log \frac{1 - P_0}{T_e(\sigma_1)}}_{\Delta = \text{const}} - \log \left( e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_1^2}} + \underbrace{\frac{P_0}{1 - P_0} \frac{T_e(\sigma_1)}{T_e(\sigma_0)}}_{t \approx \text{const}} e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_0^2}} \right) \right] = \\ &= mn \Delta + \sum_{i=1}^m \sum_{j=1}^n \underbrace{-\log \left( e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_1^2}} + t \right)}_{\hat{V}(e_{ij})} \quad (26) \end{aligned}$$

- note we are summing over all  $mn$  matches ( $m, n$  are constant!)
- when  $t = 0$  we have quadratic error function  $\hat{V}(e_{ij}) = e_{ij}^2(\mathbf{F}) / (2\sigma_1^2)$

# ► The Action of the Robust Matching Model on Data

Example for  $\hat{V}(e_{ij})$  from (26):

$$\sigma_1 = 1$$



red – the (non-robust) quadratic error

$\hat{V}(e_{ij})$  when  $t = 0$

blue – the rejected match penalty  $t$

green – robust  $\hat{V}(e_{ij})$  from (26)

- if the error of a correspondence exceeds a limit, it is ignored
- then  $\hat{V}(e_{ij}) = \text{const}$  and we just count outliers in (26)
- $t$  controls the ‘turn-off’ point
- the inlier/outlier threshold is  $e_T$  – the error for which  $(1 - P_0) p_1(e_T) = P_0 p_0(e_T)$ : note that  $t \approx 0$

$$e_T = \sigma_1 \sqrt{-\log t^2}, \quad t = e^{-\frac{1}{2} \left( \frac{e_T}{\sigma_1} \right)^2} \quad \text{e.g. } e_T = 4\sigma_1 \quad (27)$$

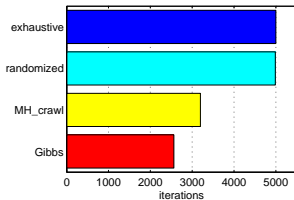
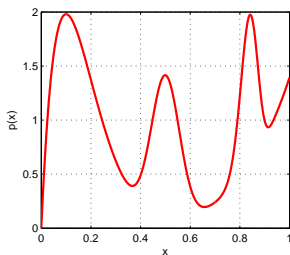
$$t = e^{\frac{-1}{2} 16} = 3.7 \cdot 10^{-4}$$

The full optimization problem (23) uses (26):

$$\mathbf{F}^* = \arg \max_{\mathbf{F}} \underbrace{\frac{p(E, D | \mathbf{F}) \cdot p(\mathbf{F})}{p(E, D)}}_{\substack{\text{data model} \quad \text{prior} \\ \text{evidence}}} \approx \arg \min_{\mathbf{F}} \left[ V(\mathbf{F}) + \sum_{i=1}^m \sum_{j=1}^n \log \left( e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_1^2}} + t \right) \right]$$

- $\pi(\mathbf{F})$  – a shorthand for the argument of the maximization
- typically we take  $V(\mathbf{F}) = -\log p(\mathbf{F}) = 0$  unless we need to stabilize a computation, e.g. when video camera moves smoothly (on a high-mass vehicle) and we have a prediction for  $\mathbf{F}$
- evidence is not needed unless we want to compare different models (e.g. homography vs. epipolar geometry)

# How To Find the Global Maxima (Modes) of a PDF?



- averaged over  $10^4$  trials
- number of proposals before  $|x - x_{\text{true}}| \leq \text{step}$

• given the function  $p(x)$  at left  
consider several methods:

p.d.f. on  $[0, 1]$ , mode at 0.1

## 1. exhaustive search

```
step = 1/(iterations-1);  
for x = 0:step:1  
    if p(x) > bestp  
        bestx = x; bestp = p(x);  
    end  
end
```

- slow algorithm (definite quantization)
- fast to implement

## 2. randomized search with uniform sampling

```
while t < iterations  
    x = rand(1);  
    if p(x) > bestp  
        bestx = x; bestp = p(x);  
    end  
    t = t+1; % time  
end
```

- equally slow algorithm
- fast to implement

*target distribution*

## 3. random sampling from $p(x)$ (Gibbs sampler)

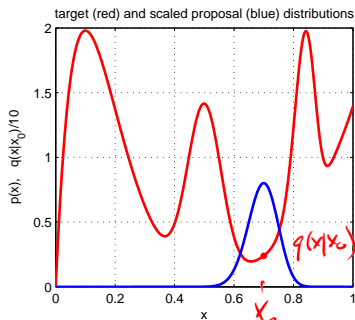
- faster algorithm
- fast to implement but often infeasible (e.g. when  $p(x)$  is data dependent (our case in correspondence prob.))

## 4. Metropolis-Hastings sampling

- almost as fast (with care)
- not so fast to implement
- rarely infeasible
- RANSAC belongs here



# How To Generate Random Samples from a Complex Distribution?



- red: probability density function  $\pi(x)$  of the toy distribution on the unit interval **target distribution**

$$\pi(x) = \sum_{i=1}^4 \gamma_i \text{Be}(x; \alpha_i, \beta_i), \quad \sum_{i=1}^4 \gamma_i = 1, \quad \gamma_i \geq 0$$

$$\text{Be}(x; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} \cdot x^{\alpha-1} (1-x)^{\beta-1}$$

- alg. for generating samples from  $\text{Be}(x; \alpha, \beta)$  is known
- $\Rightarrow$  we can generate samples from  $\pi(x)$  **how?**
- suppose we cannot sample from  $\pi(x)$  but we can sample from some 'simple' proposal distribution  $q(x | x_0)$ , given the previous sample  $x_0$  (blue)

$$q(x | x_0) = \begin{cases} U_{0,1}(x) & \text{(independent) uniform sampling} \\ \text{Be}(x; \frac{x_0}{T} + 1, \frac{1-x_0}{T} + 1) & \text{'beta' diffusion (crawler) } T - \text{temperature} \\ \pi(x) & \text{(independent) Gibbs sampler} \end{cases}$$

- note we have unified all the random sampling methods from the previous slide
- how to redistribute proposal samples  $q(x | x_0)$  to target distribution  $\pi(x)$  samples?

## ► Metropolis-Hastings (MH) Sampling

$C, S$  – configurations (of all variable values) e.g.  $C = x$  and  $\pi(C) = \pi(x)$  from →117

**Goal:** Generate a sequence of random samples  $\{C_t\}$  from target distribution  $\pi(C)$

- setup a Markov chain with a suitable transition probability to generate the sequence

### Sampling procedure

1. given current config.  $C_t$ , draw a random config. sample  $S$  from  $q(S | C_t)$   
e.g.  $q$  may use some information from  $C_t$  (Hastings)  
the evidence term drops out
2. compute acceptance probability

$$a = \min \left\{ 1, \frac{\pi(S)}{\pi(C_t)} \cdot \frac{q(C_t | S)}{q(S | C_t)} \right\}$$

3. draw a random number  $u$  from unit-interval uniform distribution  $U_{0,1}$
4. if  $u \leq a$  then  $C_{t+1} := S$  else  $C_{t+1} := C_t$

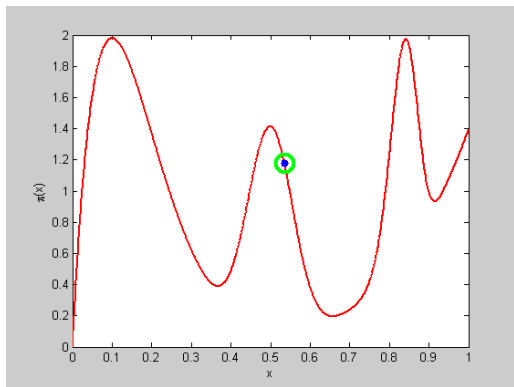
### 'Programming' an MH sampler

1. design a proposal distribution (mixture)  $q$  and a sampler from  $q$
2. write functions  $q(C_t | S)$  and  $q(S | C_t)$  that are proper distributions not always simple

### Finding the mode

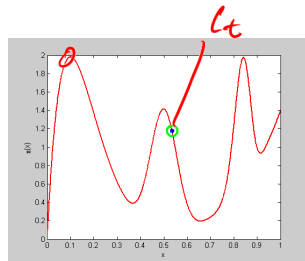
- remember the best sample fast implementation but must wait long to hit the mode
- use simulated annealing very slow
- start local optimization from the best sample good trade-off between speed and accuracy  
an optimal algorithm does not use just the best sample: a Stochastic EM Algorithm (e.g. SAEM)

# MH Sampling Demo

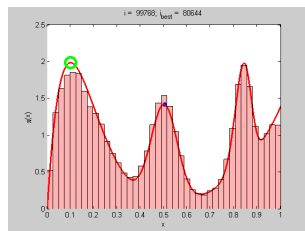


sampling process (video, 7:33, 100k samples)

- blue point: current sample
- green circle: best sample so far quality =  $\pi(x)$
- histogram: current distribution of visited states
- the vicinity of modes are the most often visited states



initial sample



final distribution of visited states

# Demo Source Code (Matlab)

```
function x = proposal_gen(x0)
% proposal generator q(x | x0)

T = 0.01; % temperature
x = betarnd(x0/T+1,(1-x0)/T+1);
end

function p = proposal_q(x, x0)
% proposal distribution q(x | x0)

T = 0.01;
p = betapdf(x, x0/T+1, (1-x0)/T+1);
end

function p = target_p(x)
% target distribution p(x)

% shape parameters:
a = [2 40 100 6];
b = [10 40 20 1];

% mixing coefficients:
w = [1 0.4 0.253 0.50]; w = w/sum(w);
p = 0;
for i = 1:length(a)
    p = p + w(i)*betapdf(x,a(i),b(i));
end
end
```

```
%% DEMO script

k = 10000; % number of samples
X = NaN(1,k); % list of samples

x0 = proposal_gen(0.5);
for i = 1:k
    x1 = proposal_gen(x0);
    a = target_p(x1)/target_p(x0) * ...
        proposal_q(x0,x1)/proposal_q(x1,x0);
    if rand(1) < a
        X(i) = x1; x0 = x1;
    else
        X(i) = x0;
    end
end

figure(1)
x = 0:0.001:1;
plot(x, target_p(x), 'r', 'linewidth',2);
hold on
binw = 0.025; % histogram bin width
n = histc(X, 0:binw:1);
h = bar(0:binw:1, n/sum(n)/binw, 'histc');
set(h, 'facecolor', 'r', 'facealpha', 0.3)
xlim([0 1]); ylim([0 2.5])
xlabel 'x'
ylabel 'p(x)'
title 'MH demo'
hold off
```

Thank You