3D Computer Vision

Radim Šára Martin Matoušek

Center for Machine Perception
Department of Cybernetics
Faculty of Electrical Engineering
Czech Technical University in Prague

https://cw.fel.cvut.cz/wiki/courses/tdv/start

http://cmp.felk.cvut.cz
mailto:sara@cmp.felk.cvut.cz
phone ext. 7203

rev. September 29, 2020

Open Informatics Master's Course

Module II

Perspective Camera

covered by

[H&Z] Secs: 2.1, 2.2, 3.1, 6.1, 6.2, 8.6, 2.5, Example: 2.19

▶Basic Geometric Entities, their Representation, and Notation

- entities have names and representations
- names and their components:

entity	in 2-space	in 3-space
point	$m = (u, v)$	$X = (x, y, z)$
line	n	O
plane		π, φ

- associated vector representations

$$\mathbf{m} = \begin{bmatrix} u \\ v \end{bmatrix} = [u, v]^\top, \quad \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{n}$$

will also be written in an 'in-line' form as $\mathbf{m} = (u, v)$, $\mathbf{X} = (x, y, z)$, etc.

- vectors are always meant to be columns $\mathbf{x} \in \mathbb{R}^{n \times 1}$
- associated homogeneous representations

$$\underline{\mathbf{m}} = [m_1, m_2, m_3]^\top, \quad \underline{\mathbf{X}} = [x_1, x_2, x_3, x_4]^\top, \quad \underline{\mathbf{n}}$$

'in-line' forms: $\underline{\mathbf{m}} = (m_1, m_2, m_3)$, $\underline{\mathbf{X}} = (x_1, x_2, x_3, x_4)$, etc.

- matrices are $\mathbf{Q} \in \mathbb{R}^{m \times n}$, linear map of a $\mathbb{R}^{n \times 1}$ vector is $\mathbf{y} = \mathbf{Qx}$
- j-th element of vector $\mathbf{m}_i$ is $(\mathbf{m}_i)_j$; element i, j of matrix $\mathbf{P}$ is $\mathbf{P}_{ij}$

▶Image Line (in 2D)

a finite line in the 2D (u, v) plane $\quad a\,u + b\,v + c = 0$

has a parameter (homogeneous) vector $\quad \underline{\mathbf{n}} \simeq (a,\ b,\ c)$, $\quad \|\underline{\mathbf{n}}\| \neq 0$

and there is an equivalence class for $\lambda \in \mathbb{R}$, $\lambda \neq 0 \quad (\lambda a,\ \lambda b,\ \lambda c) \simeq (a,\ b,\ c)$

'Finite' lines

- standard representative for finite $\underline{\mathbf{n}} = (n_1, n_2, n_3)$ is $\lambda\underline{\mathbf{n}}$, where $\lambda = \frac{\mathbf{1}}{\sqrt{n_1^2+n_2^2}}$

  assuming $n_1^2 + n_2^2 \neq 0$; $\mathbf{1}$ is the unit, usually $\mathbf{1} = 1$

'Infinite' line

- we augment the set of lines for a special entity called the **line at infinity** (ideal line)

$$\underline{\mathbf{n}}_\infty \simeq (0, 0, 1) \qquad \text{(standard representative)}$$

- the set of equivalence classes of vectors in $\mathbb{R}^3 \setminus (0, 0, 0)$ forms the projective space $\mathbb{P}^2$

  a set of rays →21

- line at infinity is a proper member of $\mathbb{P}^2$
- I may sometimes wrongly use $=$ instead of $\simeq$, if you are in doubt, ask me

▶Image Point

Finite point $\mathbf{m} = (u, v)$ is incident on a finite line $\underline{\mathbf{n}} = (a, b, c)$ iff   iff = works either way!

$$a\,u + b\,v + c = 0$$

can be rewritten as (with scalar product): $(u, v, \mathbf{1}) \cdot (a, b, c) = \underline{\mathbf{m}}^\top \underline{\mathbf{n}} = 0$

'Finite' points

- a finite point is also represented by a homogeneous vector $\underline{\mathbf{m}} \simeq (u, v, \mathbf{1})$, $\|\underline{\mathbf{m}}\| \neq 0$
- the equivalence class for $\lambda \in \mathbb{R}, \lambda \neq 0$ is $(m_1, m_2, m_3) = \lambda\,\underline{\mathbf{m}} \simeq \underline{\mathbf{m}}$
- the standard representative for finite point $\underline{\mathbf{m}}$ is $\lambda\,\underline{\mathbf{m}}$, where $\lambda = \frac{\mathbf{1}}{m_3}$ assuming $m_3 \neq 0$
- when $\mathbf{1} = 1$ then units are pixels and $\lambda\underline{\mathbf{m}} = (u, v, 1)$
- when $\mathbf{1} = f$ then all elements have a similar magnitude, $f \sim$ image diagonal

use $\mathbf{1} = 1$ unless you know what you are doing;
all entities participating in a formula must be expressed in the same units

'Infinite' points

- we augment for **points at infinity** (ideal points) $\underline{\mathbf{m}}_\infty \simeq (m_1, m_2, 0)$

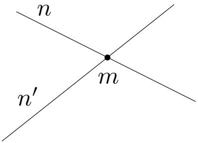
  proper members of $\mathbb{P}^2$

- all such points lie on the line at infinity (ideal line) $\underline{\mathbf{n}}_\infty \simeq (0, 0, 1)$, i.e. $\underline{\mathbf{m}}_\infty^\top \underline{\mathbf{n}}_\infty = 0$

►Line Intersection and Point Join

The point of **intersection** m of image lines n and n', $n \not\simeq n'$ is

$$\underline{\mathbf{m}} \simeq \underline{\mathbf{n}} \times \underline{\mathbf{n}}'$$

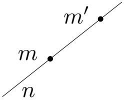


proof: If $\underline{\mathbf{m}} = \underline{\mathbf{n}} \times \underline{\mathbf{n}}'$ is the intersection point, it must be incident on both lines. Indeed, using known equivalences from vector algebra

$$\underline{\mathbf{n}}^\top \underbrace{(\underline{\mathbf{n}} \times \underline{\mathbf{n}}')}_{\underline{\mathbf{m}}} \equiv \underline{\mathbf{n}}'^\top \underbrace{(\underline{\mathbf{n}} \times \underline{\mathbf{n}}')}_{\underline{\mathbf{m}}} \equiv 0$$

The **join** n of two image points m and m', $m \not\simeq m'$ is

$$\underline{\mathbf{n}} \simeq \underline{\mathbf{m}} \times \underline{\mathbf{m}}'$$

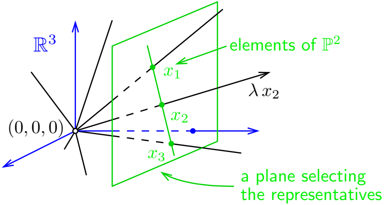


Paralel lines intersect (somewhere) on the line at infinity $\underline{\mathbf{n}}_\infty \simeq (0,0,1)$:

$$a\,u + b\,v + c = 0,$$
$$a\,u + b\,v + d = 0, \qquad d \neq c$$
$$(a,b,c) \times (a,b,d) \simeq (b,-a,0)$$

- all such intersections lie on $\underline{\mathbf{n}}_\infty$
- line at infinity therefore represents the set of (unoriented) directions in the plane
- Matlab: `m = cross(n, n_prime);`

▶Homography in $\mathbb{P}^2$



Projective plane $\mathbb{P}^2$: Vector space of dimension 3 excluding the zero vector, $\mathbb{R}^3 \setminus (0,0,0)$, factorized to linear equivalence classes ('rays'), $\underline{\mathbf{x}} \simeq \lambda \underline{\mathbf{x}}$, $\lambda \neq 0$

including 'points at infinity'

we call $\underline{\mathbf{x}} \in \mathbb{P}^2$ 'points'

Homography in $\mathbb{P}^2$: Non-singular linear mapping in $\mathbb{P}^2$ — an analogic definition for $\mathbb{P}^3$

$$\underline{\mathbf{x}}' \simeq \mathbf{H}\,\underline{\mathbf{x}}, \qquad \mathbf{H} \in \mathbb{R}^{3,3} \text{ non-singular}$$

Defining properties

- collinear points are mapped to collinear points

  lines of points are mapped to lines of points

- concurrent lines are mapped to concurrent lines

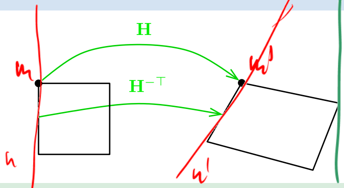
  concurrent = intersecting at a point

- and point-line incidence is preserved

  e.g. line intersection points mapped to line intersection points

- $\mathbf{H}$ is a 3×3 non-singular matrix, $\lambda\,\mathbf{H} \simeq \mathbf{H}$ equivalence class, 8 degrees of freedom
- homogeneous matrix representant: $\det \mathbf{H} = 1$
- what we call homography here is often called 'projective collineation' in mathematics

►Mapping 2D Points and Lines by Homography



$$\underline{\mathbf{m}}' \simeq \mathbf{H}\,\underline{\mathbf{m}} \qquad \text{(image) point}$$

$$\underline{\mathbf{n}}' \simeq \mathbf{H}^{-\top}\underline{\mathbf{n}} \qquad \text{(image) line} \qquad \mathbf{H}^{-\top} = (\mathbf{H}^{-1})^{\top} = (\mathbf{H}^{\top})^{-1}$$

- incidence is preserved: $(\underline{\mathbf{m}}')^{\top}\underline{\mathbf{n}}' \simeq \underline{\mathbf{m}}^{\top}\mathbf{H}^{\top}\mathbf{H}^{-\top}\underline{\mathbf{n}} = \underline{\mathbf{m}}^{\top}\underline{\mathbf{n}} = 0$

Mapping a finite 2D point $\mathbf{m} = (u, v)$ to $\underline{\mathbf{m}} = (u', v')$

1. extend the Cartesian (pixel) coordinates to homogeneous coordinates, $\underline{\mathbf{m}} = (u, v, \mathbf{1})$
2. map by homography, $\underline{\mathbf{m}}' = \mathbf{H}\,\underline{\mathbf{m}}$
3. if $m_3' \neq 0$ convert the result $\underline{\mathbf{m}}' = (m_1', m_2', m_3')$ back to Cartesian coordinates (pixels),

$$u' = \frac{m_1'}{m_3'}\mathbf{1}, \qquad v' = \frac{m_2'}{m_3'}\mathbf{1}$$

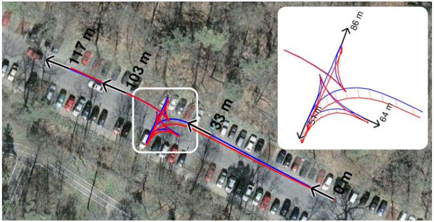
- note that, typically, $m_3' \neq 1$ $\qquad$ $m_3' = 1$ when $\mathbf{H}$ is affine
- an infinite point $\underline{\mathbf{m}} = (u, v, 0)$ maps the same way

Some Homographic Tasters

Rectification of camera rotation: →59 (geometry), →127 (homography estimation)

$\mathbf{H} \simeq \mathbf{K}\mathbf{R}^\top\mathbf{K}^{-1}$ maps from image plane to facade plane

Homographic Mouse for Visual Odometry: [Mallis 2007]



illustrations courtesy of AMSL Racing Team, Meiji University and LIBVISO: Library for VISual Odometry

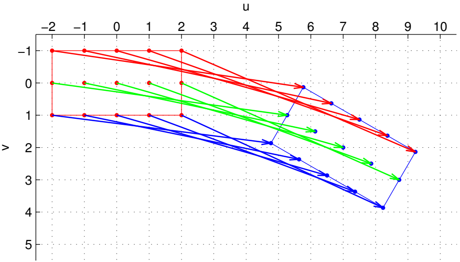
$$\mathbf{H} \simeq \mathbf{K}\left(\mathbf{R} - \frac{\mathbf{t}\mathbf{n}^\top}{d}\right)\mathbf{K}^{-1}$$ [H&Z, p. 327]

▶Homography Subgroups: Euclidean Mapping (aka Rigid Motion)

- Euclidean mapping (EM): rotation, translation and their combination

$$\mathbf{H} = \begin{bmatrix} \cos\phi & -\sin\phi & t_x \\ \sin\phi & \cos\phi & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

- eigenvalues $\left(1,\, e^{-i\phi},\, e^{i\phi}\right)$



rotation by 30°, then translation by $(7,\, 2)$

EM = The most general homography preserving

1. **areas**: $\det \mathbf{H} = 1 \;\Rightarrow\;$ unit Jacobian
2. **lengths**: Let $\underline{\mathbf{x}}_i' = \mathbf{H}\underline{\mathbf{x}}_i$ (check we can use $=$ instead of $\simeq$). Let $(x_i)_3 = 1$, Then
$$\|\underline{\mathbf{x}}_2' - \underline{\mathbf{x}}_1'\| = \|\mathbf{H}\underline{\mathbf{x}}_2 - \mathbf{H}\underline{\mathbf{x}}_1\| = \|\mathbf{H}(\underline{\mathbf{x}}_2 - \underline{\mathbf{x}}_1)\| = \cdots = \|\underline{\mathbf{x}}_2 - \underline{\mathbf{x}}_1\|$$
3. **angles** check the dot-product of normalized differences from a point $(\mathbf{x} - \mathbf{z})^\top(\mathbf{y} - \mathbf{z})$ (Cartesian(!))

- eigenvectors when $\phi \neq k\pi$, $k = 0, 1, \ldots$ (columnwise)

$$\mathbf{e}_1 \simeq \begin{bmatrix} t_x + t_y \cot\frac{\phi}{2} \\ t_y - t_x \cot\frac{\phi}{2} \\ 2 \end{bmatrix}, \quad \mathbf{e}_2 \simeq \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 \simeq \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix}$$

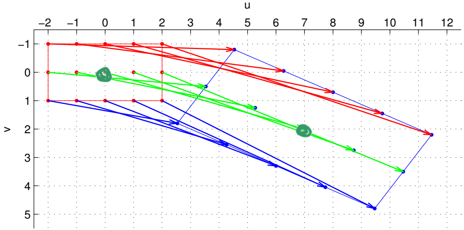
$\mathbf{e}_2, \mathbf{e}_3$ – circular points, i – imaginary unit

4. **circular points**: points at infinity $(i, 1, 0)$, $(-i, 1, 0)$ (preserved even by similarity)

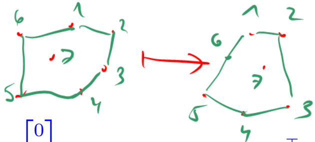
- **similarity**: scaled Euclidean mapping (does not preserve lengths, areas)

▶Homography Subgroups: Affine Mapping

$$\mathbf{H} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



rotation by 30°
then scaling by $\mathrm{diag}(1,\ 1.5,\ 1)$
then translation by $(7,\ 2)$



AM = The most general homography preserving

- parallelism
- ratio of areas
- ratio of lengths on parallel lines
- linear combinations of vectors (e.g. midpoints)
- convex hull
- line at infinity $\underline{\mathbf{n}}_\infty$ (not pointwise)

observe $\mathbf{H}^\top \underline{\mathbf{n}}_\infty \simeq \begin{bmatrix} a_{11} & a_{21} & 0 \\ a_{12} & a_{22} & 0 \\ t_x & t_y & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underline{\mathbf{n}}_\infty \quad \Rightarrow \quad \underline{\mathbf{n}}_\infty \simeq \mathbf{H}^{-\top} \underline{\mathbf{n}}_\infty$

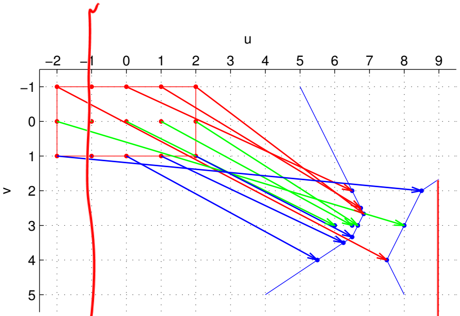
does not preserve

- lengths
- angles
- areas
- circular points

Euclidean mappings preserve all properties affine mappings preserve, of course

▶Homography Subgroups: General Homography

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$



$$\mathbf{H} = \begin{bmatrix} 7 & -0.5 & 6 \\ 3 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$$

line $\underline{\mathbf{n}} = (1, 0, 1)$ is mapped to $\underline{\mathbf{n}}_\infty$: $\mathbf{H}^{-\top}\underline{\mathbf{n}} \simeq \underline{\mathbf{n}}_\infty$

(where in the picture is the line n?)

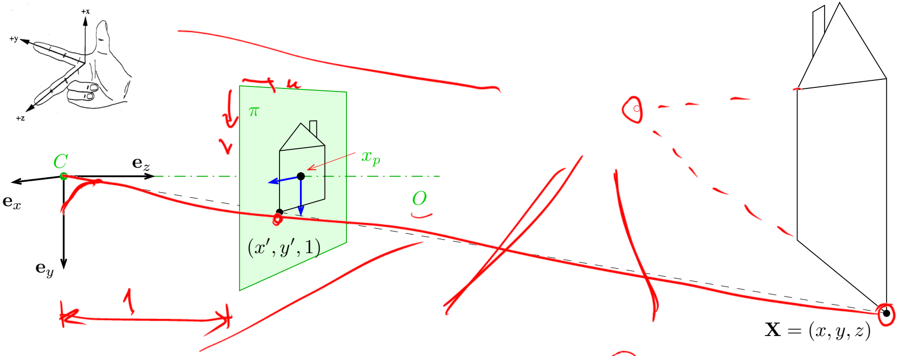
preserves only

- incidence and concurrency
- collinearity
- cross-ratio on the line →46

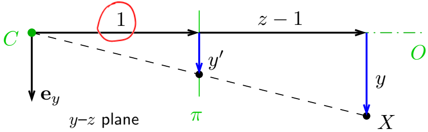
does not preserve

- lengths
- areas
- parallelism
- ratio of areas
- ratio of lengths
- linear combinations of vectors (midpoints, etc.)
- convex hull
- line at infinity $\underline{\mathbf{n}}_\infty$

▶Canonical Perspective Camera (Pinhole Camera, Camera Obscura)



1. in this picture we are looking 'down the street'
2. right-handed canonical coordinate system (x, y, z) with unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$
3. origin = center of projection C
4. image plane π at unit distance from C
5. optical axis O is perpendicular to π
6. principal point x_p: intersection of O and π
7. perspective camera is given by C and π



projected point in the natural image coordinate system:

$$\frac{y'}{1} = y' = \frac{y}{1 + z - 1} = \frac{y}{z}, \qquad x' = \frac{x}{z}$$

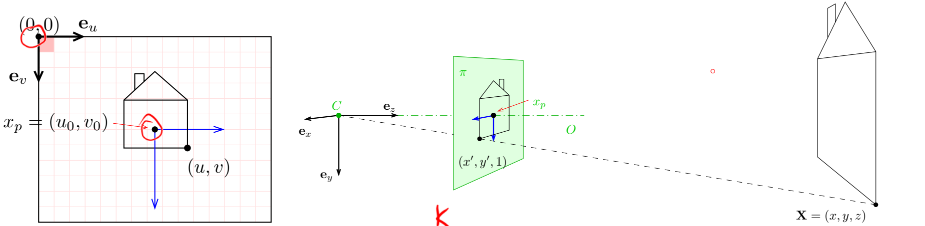
▶Natural and Canonical Image Coordinate Systems

projected point **in canonical camera** $(z \neq 0)$

$$(x', y', 1) = \left(\frac{x}{z}, \frac{y}{z}, 1\right) = \frac{1}{z}(x, y, z) \simeq \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{P}_0 = \begin{bmatrix}\mathbf{I} & \mathbf{0}\end{bmatrix}} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathbf{P}_0 \underline{\mathbf{X}}$$

projected point **in scanned image**

scale by f and translate by $(-u_0, -v_0)$



$$u = f\,\frac{x}{z} + u_0 \qquad v = f\,\frac{y}{z} + v_0$$

$$\frac{1}{z}\begin{bmatrix} f\,x + z\,u_0 \\ f\,y + z\,v_0 \\ z \end{bmatrix} \simeq \begin{bmatrix} f & 0 & u_0 \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathbf{K}\mathbf{P}_0\,\underline{\mathbf{X}} = \mathbf{P}\,\underline{\mathbf{X}}$$

- ‘calibration’ matrix $\mathbf{K}$ transforms canonical $\mathbf{P}_0$ to standard perspective camera $\mathbf{P}$

▶Computing with Perspective Camera Projection Matrix

$$\underbrace{\begin{bmatrix} f & 0 & u_0 & 0 \\ 0 & f & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{P}} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} f x + u_0 z \\ f y + v_0 z \\ z \end{bmatrix} \simeq \underbrace{\begin{bmatrix} x + \frac{z}{f} u_0 \\ y + \frac{z}{f} v_0 \\ \frac{z}{f} \end{bmatrix}}_{\text{(a)}} \simeq \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \underline{\mathbf{m}}$$

$$\frac{m_1}{m_3} = \frac{f\,x}{z} + u_0 = u, \qquad \frac{m_2}{m_3} = \frac{f\,y}{z} + v_0 = v \quad \text{when} \quad m_3 \neq 0$$

f – 'focal length' – converts length ratios to pixels, $[f] = \mathrm{px}$, $f > 0$

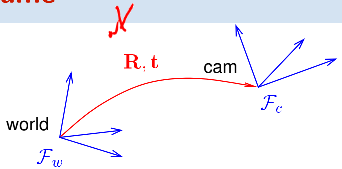
(u_0, v_0) – principal point in pixels

Perspective Camera:

1. dimension reduction — since $\mathbf{P} \in \mathbb{R}^{3,4}$
2. nonlinear unit change $\mathbf{1} \mapsto \mathbf{1} \cdot z/f$, see (a)
   for convenience we use $P_{11} = P_{22} = f$ rather than $P_{33} = 1/f$ and the u_0, v_0 in relative units
3. $m_3 = 0$ represents points at infinity in image plane π — i.e. points with $z = 0$

▶Changing The Outer (World) Reference Frame

A transformation of a point from the world to camera coordinate system:



$$\mathbf{X}_c = \mathbf{R}\,\mathbf{X}_w + \mathbf{t}$$

$\mathbf{R}$ – camera rotation matrix — world orientation in the camera coordinate frame $\mathcal{F}_c$

$\mathbf{t}$ – camera translation vector — world origin in the camera coordinate frame $\mathcal{F}_c$

$$\mathbf{P}\,\underline{\mathbf{X}}_c = \mathbf{K}\mathbf{P}_0 \begin{bmatrix}\mathbf{X}_c \\ 1\end{bmatrix} = \mathbf{K}\mathbf{P}_0 \begin{bmatrix}\mathbf{R}\mathbf{X}_w + \mathbf{t} \\ 1\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{I} & \mathbf{0}\end{bmatrix} \underbrace{\begin{bmatrix}\mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1\end{bmatrix}}_{\mathbf{T}} \begin{bmatrix}\mathbf{X}_w \\ 1\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}\underline{\mathbf{X}}_w$$

$\mathbf{P}_0$ (a 3×4 mtx) discards the last row of $\mathbf{T}$

- $\mathbf{R}$ is rotation, $\mathbf{R}^\top\mathbf{R} = \mathbf{I}$, $\det\mathbf{R} = +1$ — $\mathbf{I} \in \mathbb{R}^{3,3}$ identity matrix
- 6 **extrinsic parameters**: 3 rotation angles (Euler theorem), 3 translation components
- alternative, often used, camera representations

$$\mathbf{P} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} = \mathbf{K}\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix}$$

$\mathbf{C}$ – camera position in the world reference frame $\mathcal{F}_w$ — $\mathbf{t} = -\mathbf{R}\mathbf{C}$

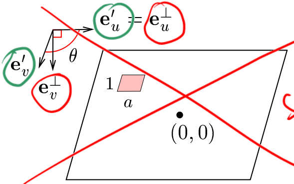
$\mathbf{r}_3^\top$ – optical axis in the world reference frame $\mathcal{F}_w$ — third row of $\mathbf{R}$: $\mathbf{r}_3 = \mathbf{R}^{-1}[0,0,1]^\top$

- we can save some conversion and computation by noting that $\mathbf{K}\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix}\underline{\mathbf{X}} = \mathbf{K}\mathbf{R}(\mathbf{X} - \mathbf{C})$

►Changing the Inner (Image) Reference Frame

The general form of calibration matrix $\mathbf{K}$ includes

- skew angle θ of the digitization raster
- pixel aspect ratio a



$$\mathbf{K} = \begin{bmatrix} a\,f & -a\,f\,\cot\theta & u_0 \\ 0 & f/\sin\theta & v_0 \\ 0 & 0 & 1 \end{bmatrix}$$

units: $[f] = \text{px}$, $[u_0] = \text{px}$, $[v_0] = \text{px}$, $[a] = 1$

⊛ H1; 2pt: Verify this $\mathbf{K}$; deadline LD+2 wk

Hints:

1. image projects to orthogonal system $F^\perp$, then it maps by skew to F', then by scale f, $a\,f$ to F'', then by translation by u_0, v_0 to F'''
2. Skew: Express point $\mathbf{x}$ as

$$\mathbf{x} = u'\mathbf{e}_{u'} + v'\mathbf{e}_{v'} = u^\perp\mathbf{e}^\perp_u + v^\perp\mathbf{e}^\perp_v$$

   $\mathbf{e}_\cdot$ are unit basis vectors
3. $\mathbf{K}$ maps from $F^\perp$ to F''' as

$$w'''\,[u''', v''', 1]^\top = \mathbf{K}[u^\perp, v^\perp, 1]^\top$$

►Summary: Projection Matrix of a General Finite Perspective Camera

$$\underline{\mathbf{m}} \simeq \mathbf{P}\underline{\mathbf{X}}, \qquad \mathbf{P} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \simeq \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} = \mathbf{K}\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix}$$

a recipe for filling $\mathbf{P}$

general finite perspective camera has 11 parameters:

- 5 intrinsic parameters: f, u_0, v_0, a, θ
- 6 extrinsic parameters: $\mathbf{t}$, $\mathbf{R}(\alpha, \beta, \gamma)$

finite camera: $\det \mathbf{K} \neq 0$

Representation Theorem: The set of projection matrices $\mathbf{P}$ of finite perspective cameras is isomorphic to the set of homogeneous 3×4 matrices with the left 3×3 submatrix $\mathbf{Q}$ non-singular.

random finite camera: `Q = rand(3,3); while det(Q)==0, Q = rand(3,3); end, P = [Q, rand(3,1)];`

▶Projection Matrix Decomposition

$$\mathbf{P} = \begin{bmatrix} \mathbf{Q} & \mathbf{q} \end{bmatrix} \longrightarrow \mathbf{K}\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}$$

$\mathbf{Q} \in \mathbb{R}^{3,3}$ full rank (if finite perspective camera; see [H&Z, Sec. 6.3] for cameras at infinity)
$\mathbf{K} \in \mathbb{R}^{3,3}$ upper triangular with positive diagonal elements
$\mathbf{R} \in \mathbb{R}^{3,3}$ rotation: $\mathbf{R}^\top\mathbf{R} = \mathbf{I}$ and $\det \mathbf{R} = +1$

1. $\begin{bmatrix} \mathbf{Q} & \mathbf{q} \end{bmatrix} = \mathbf{K}\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} = \begin{bmatrix} \mathbf{KR} & \mathbf{Kt} \end{bmatrix}$ also →35
2. RQ decomposition of $\mathbf{Q} = \mathbf{KR}$ using three Givens rotations [H&Z, p. 579]

$$\mathbf{K} = \mathbf{Q} \underbrace{\mathbf{R}_{32}\mathbf{R}_{31}\mathbf{R}_{21}}_{\mathbf{R}^{-1}} \qquad \mathbf{Q}\mathbf{R}_{32} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot \end{bmatrix}, \ \mathbf{Q}\mathbf{R}_{32}\mathbf{R}_{31} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & \cdot \end{bmatrix}, \ \mathbf{Q}\mathbf{R}_{32}\mathbf{R}_{31}\mathbf{R}_{21} = \begin{bmatrix} \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot \\ 0 & 0 & \cdot \end{bmatrix}$$

$\mathbf{R}_{ij}$ zeroes element ij in $\mathbf{Q}$ affecting only columns i and j and the sequence preserves previously zeroed elements, e.g. (see next slide for derivation details)

$$\mathbf{R}_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix} \text{ gives } \begin{matrix} c^2 + s^2 = 1 \\ 0 = k_{32} = c\, q_{32} + s\, q_{33} \end{matrix} \Rightarrow c = \frac{q_{33}}{\sqrt{q_{32}^2 + q_{33}^2}} \quad s = \frac{-q_{32}}{\sqrt{q_{32}^2 + q_{33}^2}}$$

⊛ P1; 1pt: Multiply known matrices $\mathbf{K}$, $\mathbf{R}$ and then decompose back; discuss numerical errors

- RQ decomposition nonuniqueness: $\mathbf{KR} = \mathbf{KT}^{-1}\mathbf{TR}$, where $\mathbf{T} = \operatorname{diag}(-1, -1, 1)$ is also a rotation, we must correct the result so that the diagonal elements of $\mathbf{K}$ are all positive
  'thin' RQ decomposition
- care must be taken to avoid overflow, see [Golub & van Loan 2013, sec. 5.2]

RQ Decomposition Step

```
Q = Array [q_{#1,#2} &, {3, 3}];
R32 = {{1, 0, 0}, {0, c, -s}, {0, s, c}}; R32 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix}$$

```
Q1 = Q.R32 ; Q1 // MatrixForm
```

$$\begin{pmatrix} q_{1,1} & c\, q_{1,2} + s\, q_{1,3} & -s\, q_{1,2} + c\, q_{1,3} \\ q_{2,1} & c\, q_{2,2} + s\, q_{2,3} & -s\, q_{2,2} + c\, q_{2,3} \\ q_{3,1} & c\, q_{3,2} + s\, q_{3,3} & -s\, q_{3,2} + c\, q_{3,3} \end{pmatrix}$$

```
s1 = Solve [{Q1[[3]][[2]] == 0, c^2 + s^2 == 1}, {c, s}][[2]]
```

$$\left\{ c \to \frac{q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}},\ s \to -\frac{q_{3,2}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} \right\}$$

```
Q1 /. s1 // Simplify // MatrixForm
```

$$\begin{pmatrix} q_{1,1} & \frac{-q_{1,3}\, q_{3,2} + q_{1,2}\, q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} & \frac{q_{1,2}\, q_{3,2} + q_{1,3}\, q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} \\ q_{2,1} & \frac{-q_{2,3}\, q_{3,2} + q_{2,2}\, q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} & \frac{q_{2,2}\, q_{3,2} + q_{2,3}\, q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} \\ q_{3,1} & 0 & \sqrt{q_{3,2}^2 + q_{3,3}^2} \end{pmatrix}$$

▶Center of Projection (Optical Center)

Observation: finite $\mathbf{P}$ has a non-trivial right null-space   rank 3 but 4 columns

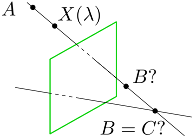
Theorem

Let $\mathbf{P}$ be a camera and let there be $\underline{\mathbf{B}} \neq \mathbf{0}$ s.t. $\mathbf{P}\,\underline{\mathbf{B}} = \mathbf{0}$. Then $\underline{\mathbf{B}}$ is equivalent to the projection center $\underline{\mathbf{C}}$ (homogeneous, in world coordinate frame).

Proof.

1. Consider spatial line AB (B is given, $A \neq B$). We can write

$$\underline{\mathbf{X}}(\lambda) \simeq \lambda\,\underline{\mathbf{A}} + (1-\lambda)\,\underline{\mathbf{B}}, \qquad \lambda \in \mathbb{R}$$



2. it projects to

$$\mathbf{P}\underline{\mathbf{X}}(\lambda) \simeq \lambda\,\mathbf{P}\,\underline{\mathbf{A}} + (1-\lambda)\,\mathbf{P}\,\underline{\mathbf{B}} \simeq \mathbf{P}\,\underline{\mathbf{A}}$$

- the entire line projects to a single point $\Rightarrow$ it must pass through the projection center of $\mathbf{P}$
- this holds for any choice of $A \neq B$ $\Rightarrow$ the only common point of the lines is the C, i.e. $\underline{\mathbf{B}} \simeq \underline{\mathbf{C}}$ □

Hence

$$\mathbf{0} = \mathbf{P}\,\underline{\mathbf{C}} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix}\begin{bmatrix}\mathbf{C}\\ 1\end{bmatrix} = \mathbf{Q}\,\mathbf{C} + \mathbf{q} \Rightarrow \boxed{\mathbf{C} = -\mathbf{Q}^{-1}\mathbf{q}}$$

$\underline{\mathbf{C}} = (c_j)$, where $c_j = (-1)^j \det \mathbf{P}^{(j)}$, in which $\mathbf{P}^{(j)}$ is $\mathbf{P}$ with column j dropped

Matlab: `C_homo = null(P);` or `C = -Q\q;`

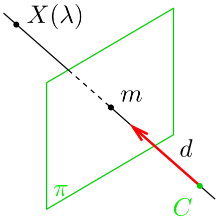
▶Optical Ray

Optical ray: Spatial line that projects to a single image point.

1. consider the following line
   $\mathbf{d}$ unit line direction vector, $\|\mathbf{d}\| = 1$, $\lambda \in \mathbb{R}$, Cartesian representation

$$\mathbf{X}(\lambda) = \mathbf{C} + \lambda\,\mathbf{d}$$

2. the projection of the (finite) point $X(\lambda)$ is



$$\underline{\mathbf{m}} \simeq \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \begin{bmatrix}\mathbf{X}(\lambda) \\ 1\end{bmatrix} = \mathbf{Q}(\mathbf{C} + \lambda\mathbf{d}) + \mathbf{q} = \lambda\,\mathbf{Q}\,\mathbf{d} =$$

$$= \lambda \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \begin{bmatrix}\mathbf{d} \\ 0\end{bmatrix}$$

…which is also the image of a point at infinity in $\mathbb{P}^3$

- optical ray line corresponding to image point m is the set

$$\mathbf{X}(\lambda) = \mathbf{C} + \mu\,\mathbf{Q}^{-1}\underline{\mathbf{m}}, \qquad \mu \in \mathbb{R}$$

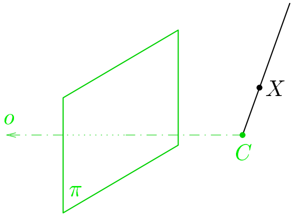
- optical ray direction may be represented by a point at infinity $(\mathbf{d}, 0)$ in $\mathbb{P}^3$
- optical ray is expressed in world coordinate frame

▶Optical Axis

Optical axis: Optical ray that is perpendicular to image plane π

1. points on a line parallel to π project to line at infinity in π:

$$\begin{bmatrix} u \\ v \\ 0 \end{bmatrix} \simeq \mathbf{P}\underline{\mathbf{X}} = \begin{bmatrix} \mathbf{q}_1^\top & q_{14} \\ \mathbf{q}_2^\top & q_{24} \\ \mathbf{q}_3^\top & q_{34} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$



2. therefore the set of points X is parallel to π iff

$$\mathbf{q}_3^\top \mathbf{X} + q_{34} = 0$$

3. this is a plane with $\pm\mathbf{q}_3$ as the normal vector
4. optical axis direction: substitution $\mathbf{P} \mapsto \lambda\mathbf{P}$ must not change the direction
5. we select (assuming $\det(\mathbf{R}) > 0$)

$$\mathbf{o} = \det(\mathbf{Q})\,\mathbf{q}_3$$

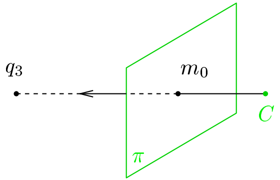
if $\mathbf{P} \mapsto \lambda\mathbf{P}$ then $\det(\mathbf{Q}) \mapsto \lambda^3 \det(\mathbf{Q})$ and $\mathbf{q}_3 \mapsto \lambda\,\mathbf{q}_3$ [H&Z, p. 161]

- the axis is expressed in world coordinate frame

▶Principal Point

Principal point: The intersection of image plane and the optical axis

1. as we saw, $\mathbf{q}_3$ is the directional vector of optical axis
2. we take point at infinity on the optical axis that must project to the principal point m_0



3. then

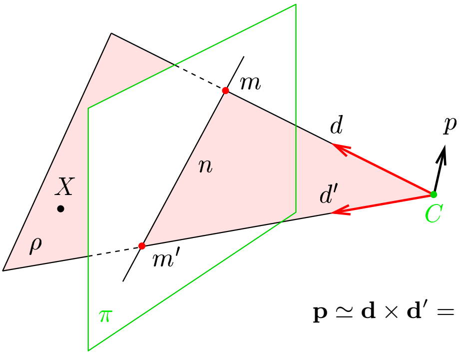
$$\underline{\mathbf{m}}_0 \simeq \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \begin{bmatrix}\mathbf{q}_3 \\ 0\end{bmatrix} = \mathbf{Q}\,\mathbf{q}_3$$

principal point: $\underline{\mathbf{m}}_0 \simeq \mathbf{Q}\,\mathbf{q}_3$

- principal point is also the center of radial distortion

▶Optical Plane

A spatial plane with normal p containing the projection center C and a given image line n.



optical ray given by m $\quad \mathbf{d} \simeq \mathbf{Q}^{-1}\underline{\mathbf{m}}$

optical ray given by m' $\quad \mathbf{d}' \simeq \mathbf{Q}^{-1}\underline{\mathbf{m}}'$

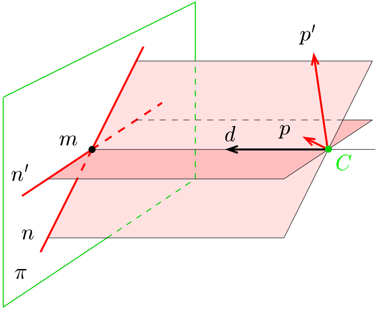
$$\mathbf{p} \simeq \mathbf{d} \times \mathbf{d}' = (\mathbf{Q}^{-1}\underline{\mathbf{m}}) \times (\mathbf{Q}^{-1}\underline{\mathbf{m}}') = \mathbf{Q}^{\top}(\underline{\mathbf{m}} \times \underline{\mathbf{m}}') = \mathbf{Q}^{\top}\underline{\mathbf{n}}$$

- note the way $\mathbf{Q}$ factors out!

hence, $0 = \mathbf{p}^{\top}(\mathbf{X} - \mathbf{C}) = \underline{\mathbf{n}}^{\top}\underbrace{\mathbf{Q}(\mathbf{X} - \mathbf{C})}_{\to 30} = \underline{\mathbf{n}}^{\top}\mathbf{P}\underline{\mathbf{X}} = (\mathbf{P}^{\top}\underline{\mathbf{n}})^{\top}\underline{\mathbf{X}}$ for every X in plane ρ

optical plane is given by n: $\quad \underline{\boldsymbol{\rho}} \simeq \mathbf{P}^{\top}\underline{\mathbf{n}}$ $\qquad \rho_1\, x + \rho_2\, y + \rho_3\, z + \rho_4 = 0$

Cross-Check: Optical Ray as Optical Plane Intersection



optical plane normal given by n $\qquad$ $\mathbf{p} = \mathbf{Q}^\top \underline{\mathbf{n}}$

optical plane normal given by n' $\qquad$ $\mathbf{p}' = \mathbf{Q}^\top \underline{\mathbf{n}}'$

$$\mathbf{d} = \mathbf{p} \times \mathbf{p}' = (\mathbf{Q}^\top \underline{\mathbf{n}}) \times (\mathbf{Q}^\top \underline{\mathbf{n}}') = \mathbf{Q}^{-1}(\underline{\mathbf{n}} \times \underline{\mathbf{n}}') = \mathbf{Q}^{-1}\underline{\underline{\mathbf{m}}}$$

▶Summary: Projection Center; Optical Ray, Axis, Plane

General (finite) camera

$$\mathbf{P} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} = \begin{bmatrix}\mathbf{q}_1^\top & q_{14} \\ \mathbf{q}_2^\top & q_{24} \\ \mathbf{q}_3^\top & q_{34}\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} = \mathbf{K}\,\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix}$$

$\underline{\mathbf{C}} \simeq \operatorname{rnull}(\mathbf{P}), \quad \mathbf{C} = -\mathbf{Q}^{-1}\mathbf{q}$ — projection center (world coords.) →35

$\mathbf{d} = \mathbf{Q}^{-1}\,\underline{\mathbf{m}}$ — optical ray direction (world coords.) →36

$\mathbf{o} = \det(\mathbf{Q})\,\mathbf{q}_3$ — outward optical axis (world coords.) →37

$\underline{\mathbf{m}}_0 \simeq \mathbf{Q}\,\mathbf{q}_3$ — principal point (in image plane) →38

$\underline{\boldsymbol{\rho}} = \mathbf{P}^\top\,\underline{\mathbf{n}}$ — optical plane (world coords.) →39

$$\mathbf{K} = \begin{bmatrix} f & -f\cot\theta & u_0 \\ 0 & f/(a\sin\theta) & v_0 \\ 0 & 0 & 1\end{bmatrix}$$ camera (calibration) matrix (f, u_0, v_0 in pixels) →31

$\mathbf{R}$ — camera rotation matrix (cam coords.) →30

$\mathbf{t}$ — camera translation vector (cam coords.) →30

What Can We Do with An ‘Uncalibrated’ Perspective Camera?

How far is the engine?

distance between sleepers (ties) 0.806m but we cannot count them, the image resolution is too low

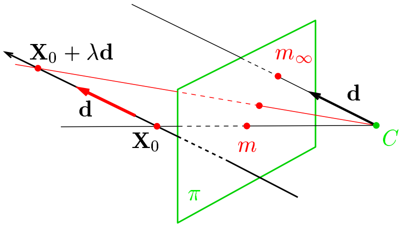
We will review some life-saving theory. . .
. . . and build a bit of geometric intuition. . .

In fact

- ‘uncalibrated’ = the image contains a calibrating object that suffices for the task at hand

►Vanishing Point

Vanishing point: the limit of the projection of a point that moves along a space line infinitely in one direction. the image of the point at infinity on the line



$$\underline{\mathbf{m}}_\infty \simeq \lim_{\lambda \to \pm\infty} \mathbf{P} \begin{bmatrix} \mathbf{X}_0 + \lambda\mathbf{d} \\ 1 \end{bmatrix} = \cdots \simeq \mathbf{Q}\,\mathbf{d}$$

⊛ P1; 1pt: Prove (use Cartesian coordinates and L'Hôpital's rule)

- the V.P. of a spatial line with directional vector $\mathbf{d}$ is $\underline{\mathbf{m}}_\infty \simeq \mathbf{Q}\,\mathbf{d}$
- V.P. is independent on line position $\mathbf{X}_0$, it depends on its directional vector only
- all parallel lines share the same V.P., including the optical ray defined by m_∞

Some Vanishing Point "Applications"

where is the sun?

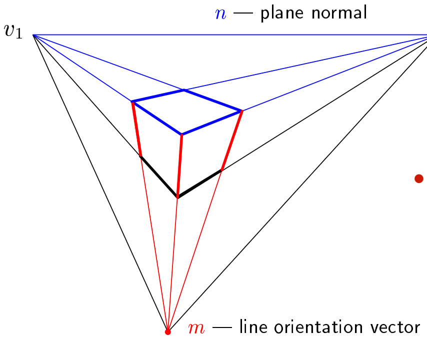
what is the wind direction?
(must have video)

fly above the lane,
at constant altitude!

▶Vanishing Line

Vanishing line: The set of vanishing points of all lines in a plane

the image of the line at infinity in the plane
and in all parallel planes



- any box with parallel edges

- V.L. n corresponds to spatial plane of normal vector $\mathbf{p} = \mathbf{Q}^\top \underline{\mathbf{n}}$

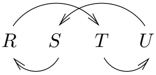
  because this is the normal vector of a parallel optical plane (!) →39

- a spatial plane of normal vector $\mathbf{p}$ has a V.L. represented by $\underline{\mathbf{n}} = \mathbf{Q}^{-\top} \mathbf{p}$.

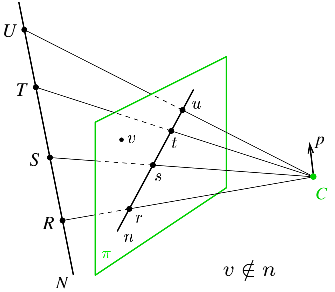
▶Cross Ratio

Four distinct finite collinear spatial points R, S, T, U define cross-ratio

$$[RSTU] = \frac{|\overrightarrow{RT}|}{|\overrightarrow{SR}|}\,\frac{|\overrightarrow{US}|}{|\overrightarrow{TU}|}$$



a mnemonic (∞)



$|\overrightarrow{RT}|$ – signed distance from R to T in the arrow direction

6 cross-ratios from four points:

$$[SRUT] = [RSTU],\ [RSUT] = \frac{1}{[RSTU]},\ [RTSU] = 1 - [RSTU],\ \cdots$$

Obs: $$[RSTU] = \frac{|\underline{\mathbf{r}}\ \underline{\mathbf{t}}\ \underline{\mathbf{v}}|}{|\underline{\mathbf{s}}\ \underline{\mathbf{r}}\ \underline{\mathbf{v}}|} \cdot \frac{|\underline{\mathbf{u}}\ \underline{\mathbf{s}}\ \underline{\mathbf{v}}|}{|\underline{\mathbf{t}}\ \underline{\mathbf{u}}\ \underline{\mathbf{v}}|}, \quad |\underline{\mathbf{r}}\ \underline{\mathbf{t}}\ \underline{\mathbf{v}}| = \det\left[\underline{\mathbf{r}}\ \underline{\mathbf{t}}\ \underline{\mathbf{v}}\right] = (\underline{\mathbf{r}} \times \underline{\mathbf{t}})^\top \underline{\mathbf{v}} \qquad (1)$$

Corollaries:

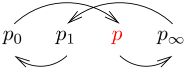
- cross ratio is invariant under homographies $\underline{\mathbf{x}}' \simeq \mathbf{H}\underline{\mathbf{x}}$ plug $\mathbf{H}\underline{\mathbf{x}}$ in (1): $(\mathbf{H}^{-\top}(\underline{\mathbf{r}} \times \underline{\mathbf{t}}))^\top \mathbf{H}\underline{\mathbf{v}}$
- cross ratio is invariant under perspective projection: $[RSTU] = [r\,s\,t\,u]$

- 4 collinear points: any perspective camera will "see" the same cross-ratio of their images
- we measure the same cross-ratio in image as on the world line
- one of the points R, S, T, U may be at infinity (we take the limit, in effect $\frac{\infty}{\infty} = 1$)

▶1D Projective Coordinates

The 1-D projective coordinate of a point P is defined by the following cross-ratio:

$$[P] = [P_0\, P_1\, P\, P_\infty] = [p_0\, p_1\, p\, p_\infty] = \frac{|\overrightarrow{p_0\, p}|}{|\overrightarrow{p_1\, p_0}|}\,\frac{|\overrightarrow{p_\infty\, p_1}|}{|\overrightarrow{p\, p_\infty}|} = [p]$$

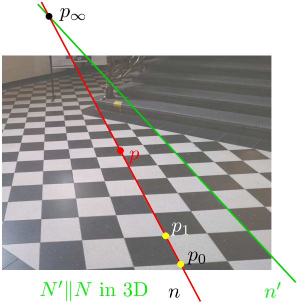


naming convention:

P_0 – the origin	$[P_0] = 0$
P_1 – the unit point	$[P_1] = 1$
P_∞ – the supporting point	$[P_\infty] = \pm\infty$

$$[P] = [p]$$

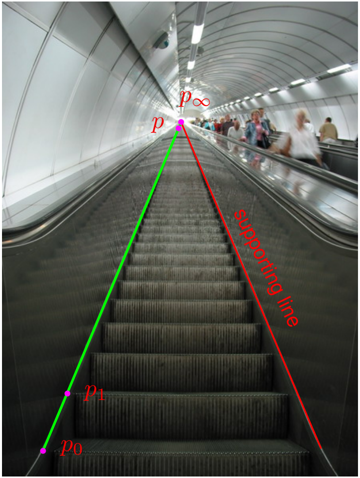
$[P]$ is equal to Euclidean coordinate along N
$[p]$ is its measurement in the image plane



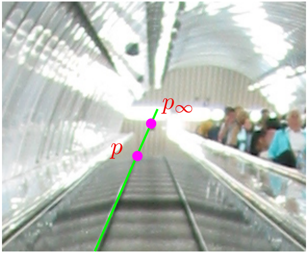
Applications

- Given the image of a 3D line N, the origin, the unit point, and the vanishing point, then the Euclidean coordinate of any point $P \in N$ can be determined →48
- Finding v.p. of a line through a regular object →49

Application: Counting Steps



- Namesti Miru underground station in Prague

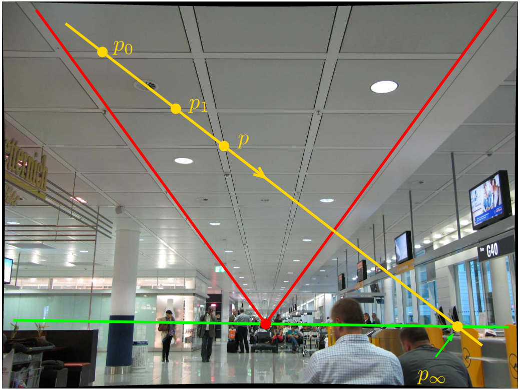


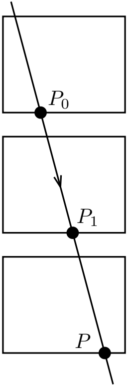
detail around the vanishing point

Result: $[P] = 214$ steps (correct answer is 216 steps)

4Mpx camera

Application: Finding the Horizon from Repetitions





in 3D: $|P_0P| = 2|P_0P_1|$ then [H&Z, p. 218]

$$[P_0 P_1 P P_\infty] = \frac{|P_0 P|}{|P_1 P_0|} = 2 \quad \Rightarrow \quad x_\infty = \frac{x_0\,(2\,x - x_1) - x\,x_1}{x + x_0 - 2\,x_1}$$

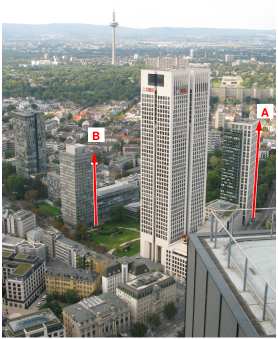
- x – 1D coordinate along the yellow line, positive in the arrow direction
- could be applied to counting steps (→48) if there was no supporting line

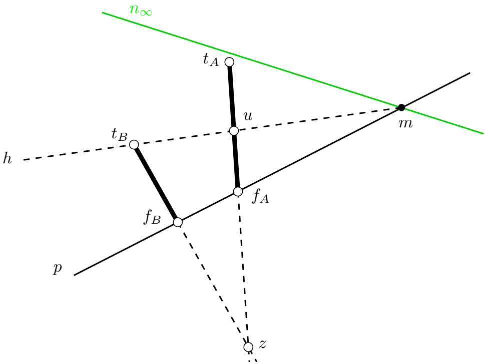
⊛ P1; 1pt: How high is the camera above the floor?

Homework Problem

⊛ H2; 3pt: What is the ratio of heights of Building A to Building B?

- expected: conceptual solution; use notation from this figure
- deadline: LD+2 weeks

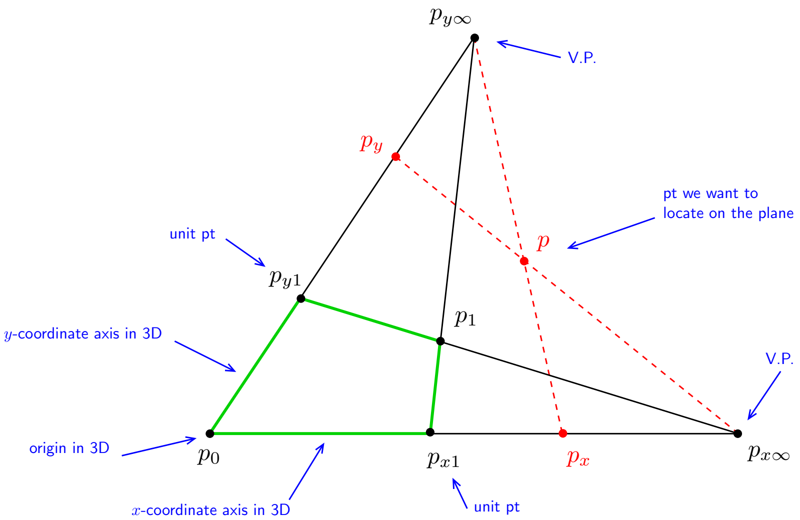




Hints

1. What are the interesting properties of line h connecting the top t_B of Buiding B with the point m at which the horizon intersects the line p joining the foots f_A, f_B of both buildings? [1 point]
2. How do we actually get the horizon n_∞? (we do not see it directly, there are some hills there. . .) [1 point]
3. Give the formula for measuring the length ratio. [formula = 1 point]

2D Projective Coordinates



$$[P_x] = [P_0\ P_{x1}\ P_x\ P_{x\infty}] \qquad [P_y] = [P_0\ P_{y1}\ P_y\ P_{y\infty}]$$

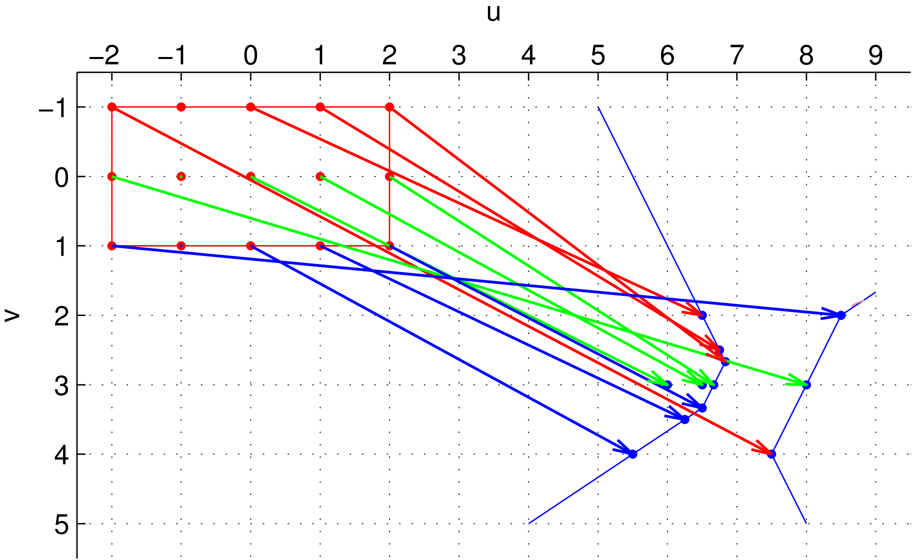
Application: Measuring on the Floor (Wall, etc)

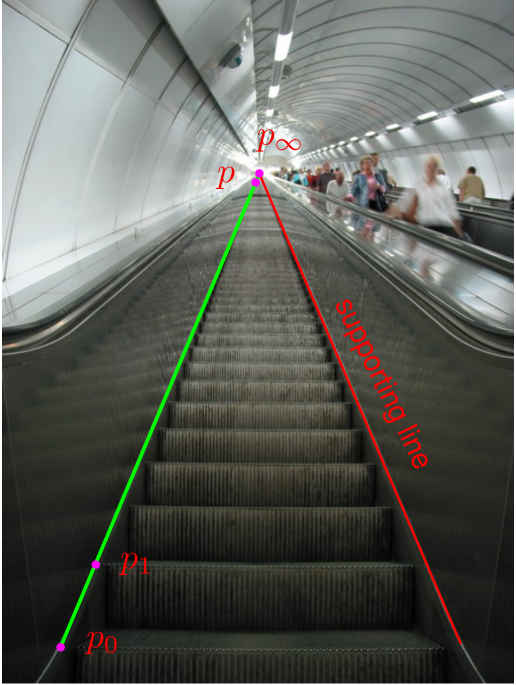
San Giovanni in Laterano, Rome

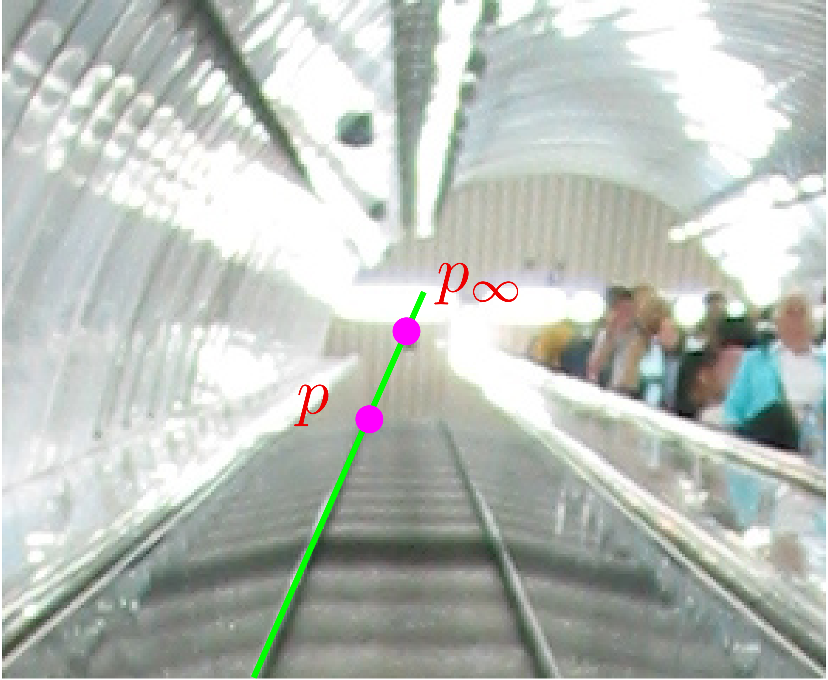
- measuring distances on the floor in terms of tile units
- what are the dimensions of the seal? Is it circular (assuming square tiles)?
- needs no explicit camera calibration

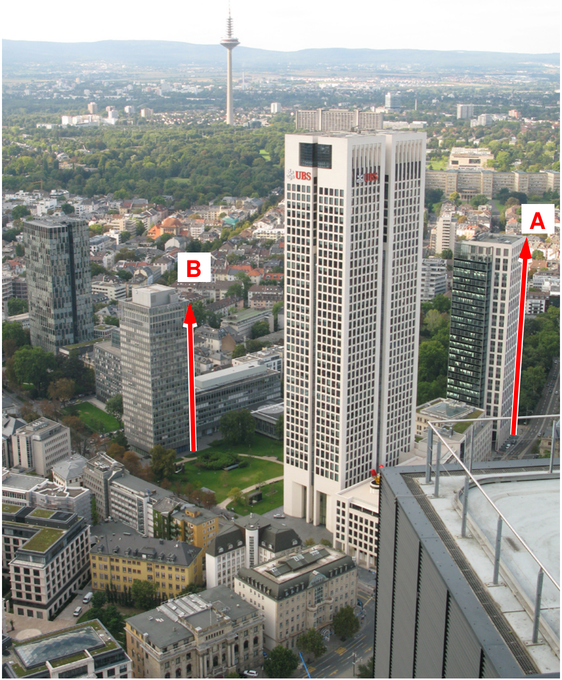
because we can see the calibrating object (vanishing points)

Thank You


u
−2
−1
0
1
2
3
4
5
6
7
8
9
v
−1
0
1
2
3
4
5


p_∞
p
supporting line
p_1
p_0


p_∞
p


UBS
A
B