3D Computer Vision

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rev. October 20, 2020

Open Informatics Master's Course

The Relative Orientation Problem

Problem: Given point triples (X_1, X_2, X_3) and (Y_1, Y_2, Y_3) in a general position in $\mathbf{R}^3$ such that the correspondence $X_i \leftrightarrow Y_i$ is known, determine the relative orientation $(\mathbf{R}, \mathbf{t})$ that maps $\mathbf{X}_i$ to $\mathbf{Y}_i$, i.e.

$$\mathbf{Y}_i = \mathbf{R}\mathbf{X}_i + \mathbf{t}, \quad i = 1, 2, 3\,.$$

Applies to:

- 3D scanners
- partial reconstructions from different viewpoints

Obs: Let the centroid be $\bar{\mathbf{X}} = \frac{1}{3}\sum_i \mathbf{X}_i$ and analogically for $\bar{\mathbf{Y}}$. Then

$$\bar{\mathbf{Y}} = \mathbf{R}\bar{\mathbf{X}} + \mathbf{t}.$$

Therefore

$$\mathbf{Z}_i \overset{\text{def}}{=} (\mathbf{Y}_i - \bar{\mathbf{Y}}) = \mathbf{R}(\mathbf{X}_i - \bar{\mathbf{X}}) \overset{\text{def}}{=} \mathbf{R}\mathbf{W}_i$$

If all dot products are equal, $\mathbf{Z}_i^\top \mathbf{Z}_j = \mathbf{W}_i^\top \mathbf{W}_j$ for $i, j = 1, 2, 3$, we have

$$\mathbf{R}^* = \begin{bmatrix}\mathbf{W}_1 & \mathbf{W}_2 & \mathbf{W}_3\end{bmatrix}^{-1} \begin{bmatrix}\mathbf{Z}_1 & \mathbf{Z}_2 & \mathbf{Z}_3\end{bmatrix}$$

Otherwise (in practice) we setup a minimization problem

$$\mathbf{R}^* = \arg\min_{\mathbf{R}} \sum_{i=1}^{3} \|\mathbf{Z}_i - \mathbf{R}\mathbf{W}_i\|^2 \quad \text{s.t.} \quad \mathbf{R}^\top\mathbf{R} = \mathbf{I}, \quad \det \mathbf{R} = 1$$

$$\arg\min_{\mathbf{R}} \sum_i \|\mathbf{Z}_i - \mathbf{R}\mathbf{W}_i\|^2 = \arg\min_{\mathbf{R}} \sum_i \Big(\|\mathbf{Z}_i\|^2 - 2\mathbf{Z}_i^\top \mathbf{R}\mathbf{W}_i + \|\mathbf{W}_i\|^2\Big) = \cdots$$

$$\cdots = \arg\max_{\mathbf{R}} \sum_i \mathbf{Z}_i^\top \mathbf{R}\mathbf{W}_i$$

cont'd (What is Linear Algebra Telling Us?)

Obs 1: Let $\mathbf{A} : \mathbf{B} = \sum_{i,j} a_{ij} b_{ij}$ be the dot-product (Frobenius inner product) over real matrices. Then

$$\mathbf{A} : \mathbf{B} = \mathbf{B} : \mathbf{A} = \operatorname{tr}(\mathbf{A}^\top \mathbf{B})$$

Obs 2: (cyclic property for matrix trace)

$$\operatorname{tr}(\mathbf{A}\mathbf{B}\mathbf{C}) = \operatorname{tr}(\mathbf{C}\mathbf{A}\mathbf{B})$$

Obs 3: ($\mathbf{Z}_i$, $\mathbf{W}_i$ are vectors)

$$\mathbf{Z}_i^\top \mathbf{R} \mathbf{W}_i = \operatorname{tr}(\mathbf{Z}_i^\top \mathbf{R} \mathbf{W}_i) = \operatorname{tr}(\mathbf{W}_i \mathbf{Z}_i^\top \mathbf{R}) = (\mathbf{Z}_i \mathbf{W}_i^\top) : \mathbf{R} = \mathbf{R} : (\mathbf{Z}_i \mathbf{W}_i^\top)$$

Let the SVD be

$$\sum_i \mathbf{Z}_i \mathbf{W}_i^\top \stackrel{\text{def}}{=} \mathbf{M} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$$

Then

$$\mathbf{R} : \mathbf{M} = \mathbf{R} : (\mathbf{U}\mathbf{D}\mathbf{V}^\top) = \operatorname{tr}(\mathbf{R}^\top \mathbf{U}\mathbf{D}\mathbf{V}^\top) = \operatorname{tr}(\mathbf{V}^\top \mathbf{R}^\top \mathbf{U}\mathbf{D}) = (\mathbf{U}^\top \mathbf{R} \mathbf{V}) : \mathbf{D}$$

We are solving

$$\mathbf{R}^* = \arg\max_{\mathbf{R}} \sum_i \mathbf{Z}_i^\top \mathbf{R} \mathbf{W}_i = \arg\max_{\mathbf{R}} \left(\mathbf{U}^\top \mathbf{R} \mathbf{V}\right) : \mathbf{D}$$

A particular solution is found as follows:

- $\mathbf{U}^\top \mathbf{R} \mathbf{V}$ must be (1) orthogonal, and most similar to (2) diagonal, (3) positive definite
- Since $\mathbf{U}$, $\mathbf{V}$ are orthogonal matrices then the solution to the problem is among $\mathbf{R}^* = \mathbf{U}\mathbf{S}\mathbf{V}^\top$, where $\mathbf{S}$ is diagonal and orthogonal, i.e. one of

$$\pm\,\mathrm{diag}(1,1,1), \quad \pm\,\mathrm{diag}(1,-1,-1), \quad \pm\,\mathrm{diag}(-1,1,-1), \quad \pm\,\mathrm{diag}(-1,-1,1)$$

- $\mathbf{U}^\top \mathbf{V}$ is not necessarily positive definite
- We choose $\mathbf{S}$ so that $(\mathbf{R}^*)^\top \mathbf{R}^* = \mathbf{I}$

Alg:

1. Compute matrix $\mathbf{M} = \sum_i \mathbf{Z}_i \mathbf{W}_i^\top$.
2. Compute SVD $\mathbf{M} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$.
3. Compute all $\mathbf{R}_k = \mathbf{U}\mathbf{S}_k\mathbf{V}^\top$ that give $\mathbf{R}_k^\top \mathbf{R}_k = \mathbf{I}$.
4. Compute $\mathbf{t}_k = \bar{\mathbf{Y}} - \mathbf{R}_k \bar{\mathbf{X}}$.

- The algorithm can be used for more than 3 points
- Triple pairs can be pre-filtered based on motion invariants (lengths, angles)
- The P3P problem is very similar but not identical

Module IV

Computing with a Camera Pair

4.1 Camera Motions Inducing Epipolar Geometry

4.2 Estimating Fundamental Matrix from 7 Correspondences

4.3 Estimating Essential Matrix from 5 Correspondences

4.4 Triangulation: 3D Point Position from a Pair of Corresponding Points

covered by

[1] [H&Z] Secs: 9.1, 9.2, 9.6, 11.1, 11.2, 11.9, 12.2, 12.3, 12.5.1

[2] H. Li and R. Hartley. Five-point motion estimation made easy. In *Proc ICPR* 2006, pp. 630–633

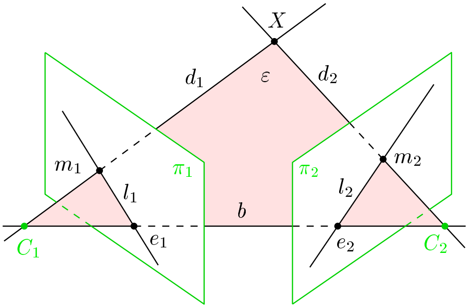
additional references

H. Longuet-Higgins. A computer algorithm for reconstructing a scene from two projections. *Nature*, 293 (5828):133–135, 1981.

▶Geometric Model of a Camera Pair

Epipolar geometry:

- brings constraints necessary for inter-image matching
- its parametric form encapsulates information about the relative pose of two cameras



two-camera setup

Description

- baseline b joins projection centers C_1, C_2

$$\mathbf{b} = \mathbf{C}_2 - \mathbf{C}_1$$

- epipole $e_i \in \pi_i$ is the image of C_j:

$$\underline{\mathbf{e}}_1 \simeq \mathbf{P}_1 \underline{\mathbf{C}}_2, \quad \underline{\mathbf{e}}_2 \simeq \mathbf{P}_2 \underline{\mathbf{C}}_1$$

- $l_i \in \pi_i$ is the image of epipolar plane

$$\varepsilon = (C_2, X, C_1)$$

- l_j is the epipolar line in image π_j induced by m_i in image π_i

Epipolar constraint: corresponding d_2, b, d_1 are coplanar

a necessary condition →87

$$\mathbf{P}_i = \begin{bmatrix} \mathbf{Q}_i & \mathbf{q}_i \end{bmatrix} = \mathbf{K}_i \begin{bmatrix} \mathbf{R}_i & \mathbf{t}_i \end{bmatrix} = \mathbf{K}_i \mathbf{R}_i \begin{bmatrix} \mathbf{I} & -\mathbf{C}_i \end{bmatrix} \qquad i = 1, 2 \qquad \rightarrow 31$$

Epipolar Geometry Example: Forward Motion

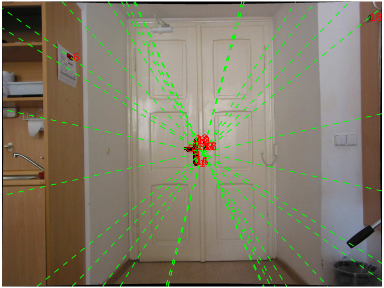


image 1

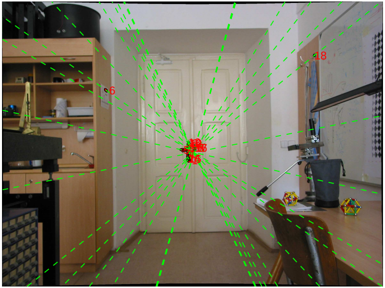
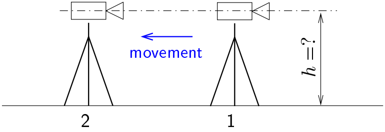


image 2

- red: correspondences
- green: epipolar line pairs per correspondence

click on the image to see their IDs
same ID in both images

How high was the camera above the floor?



▶Cross Products and Maps by Skew-Symmetric 3×3 Matrices

- There is an equivalence $\mathbf{b} \times \mathbf{m} = [\mathbf{b}]_\times \mathbf{m}$, where $[\mathbf{b}]_\times$ is a 3×3 skew-symmetric matrix

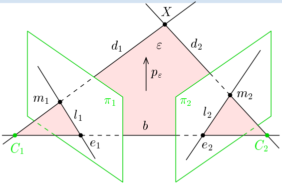
$$[\mathbf{b}]_\times = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}, \qquad \text{assuming} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Some properties

1. $[\mathbf{b}]_\times^\top = -[\mathbf{b}]_\times$ — the general antisymmetry property
2. $\mathbf{A}$ is skew-symmetric iff $\mathbf{x}^\top \mathbf{A} \mathbf{x} = 0$ for all $\mathbf{x}$ — skew-sym mtx generalizes cross products
3. $[\mathbf{b}]_\times^3 = -\|\mathbf{b}\|^2 \cdot [\mathbf{b}]_\times$
4. $\big\|[\mathbf{b}]_\times\big\|_F = \sqrt{2}\,\|\mathbf{b}\|$ — Frobenius norm ($\|\mathbf{A}\|_F = \sqrt{\operatorname{tr}(\mathbf{A}^\top \mathbf{A})} = \sqrt{\sum_{i,j} |a_{ij}|^2}$)
5. $[\mathbf{b}]_\times \mathbf{b} = \mathbf{0}$
6. $\operatorname{rank} [\mathbf{b}]_\times = 2$ iff $\|\mathbf{b}\| > 0$ — check minors of $[\mathbf{b}]_\times$
7. eigenvalues of $[\mathbf{b}]_\times$ are $(0, \lambda, -\lambda)$
8. for any 3×3 regular $\mathbf{B}$: $\mathbf{B}^\top [\mathbf{B}\mathbf{z}]_\times \mathbf{B} = \det \mathbf{B}\, [\mathbf{z}]_\times$ — follows from the factoring on →39
9. in particular: if $\mathbf{R}\mathbf{R}^\top = \mathbf{I}$ then $[\mathbf{R}\mathbf{b}]_\times = \mathbf{R}[\mathbf{b}]_\times \mathbf{R}^\top$

- note that if $\mathbf{R}_b$ is rotation about $\mathbf{b}$ then $\mathbf{R}_b \mathbf{b} = \mathbf{b}$
- note $[\mathbf{b}]_\times$ is not a homography; it is not a rotation matrix — it is the logarithm of a rotation mtx

▶Expressing Epipolar Constraint Algebraically



$$\mathbf{P}_i = \begin{bmatrix}\mathbf{Q}_i & \mathbf{q}_i\end{bmatrix} = \mathbf{K}_i \begin{bmatrix}\mathbf{R}_i & \mathbf{t}_i\end{bmatrix},\; i = 1, 2$$

$\mathbf{R}_{21}$ – relative camera rotation, $\mathbf{R}_{21} = \mathbf{R}_2\mathbf{R}_1^\top$

$\mathbf{t}_{21}$ – relative camera translation, $\mathbf{t}_{21} = \mathbf{t}_2 - \mathbf{R}_{21}\mathbf{t}_1 = -\mathbf{R}_2\mathbf{b}$ →74

$\mathbf{b}$ – baseline vector (world coordinate system)

remember: $\mathbf{C} = -\mathbf{Q}^{-1}\mathbf{q} = -\mathbf{R}^\top\mathbf{t}$ →33 and 35

$$\boxed{0 = \mathbf{d}_2^\top \underbrace{\mathbf{p}_\varepsilon}_{\text{normal of } \varepsilon} \simeq \underbrace{(\mathbf{Q}_2^{-1}\underline{\mathbf{m}}_2)^\top}_{\text{optical ray}} \underbrace{\mathbf{Q}_1^\top \underline{\mathbf{l}}_1}_{\text{optical plane}} = \underline{\mathbf{m}}_2^\top \underbrace{\mathbf{Q}_2^{-\top}\mathbf{Q}_1^\top(\underline{\mathbf{e}}_1 \times \underline{\mathbf{m}}_1)}_{\text{image of } \varepsilon \text{ in } \pi_2} = \underline{\mathbf{m}}_2^\top \underbrace{\left(\mathbf{Q}_2^{-\top}\mathbf{Q}_1^\top [\underline{\mathbf{e}}_1]_\times\right)}_{\text{fundamental matrix } \mathbf{F}} \underline{\mathbf{m}}_1}$$

Epipolar constraint $\underline{\mathbf{m}}_2^\top \mathbf{F}\, \underline{\mathbf{m}}_1 = 0$ is a point-line incidence constraint

- point $\underline{\mathbf{m}}_2$ is incident on epipolar line $\underline{\mathbf{l}}_2 \simeq \mathbf{F}\underline{\mathbf{m}}_1$
- point $\underline{\mathbf{m}}_1$ is incident on epipolar line $\underline{\mathbf{l}}_1 \simeq \mathbf{F}^\top\underline{\mathbf{m}}_2$
- $\mathbf{F}\underline{\mathbf{e}}_1 = \mathbf{F}^\top\underline{\mathbf{e}}_2 = \mathbf{0}$ (non-trivially)
- all epipolars meet at the epipole

$$\underline{\mathbf{e}}_1 \simeq \mathbf{Q}_1\mathbf{C}_2 + \mathbf{q}_1 = \mathbf{Q}_1\mathbf{C}_2 - \mathbf{Q}_1\mathbf{C}_1 = \mathbf{K}_1\mathbf{R}_1\mathbf{b} = -\mathbf{K}_1\mathbf{R}_1\mathbf{R}_2^\top\mathbf{t}_{21} = -\mathbf{K}_1\mathbf{R}_{21}^\top\mathbf{t}_{21}$$

$$\mathbf{F} = \mathbf{Q}_2^{-\top}\mathbf{Q}_1^\top[\underline{\mathbf{e}}_1]_\times = \mathbf{Q}_2^{-\top}\mathbf{Q}_1^\top[-\mathbf{K}_1\mathbf{R}_{21}^\top\mathbf{t}_{21}]_\times \overset{\circledast 1}{=} \cdots \simeq \mathbf{K}_2^{-\top}[-\mathbf{t}_{21}]_\times\mathbf{R}_{21}\mathbf{K}_1^{-1}$$ fundamental

$$\mathbf{E} = [-\mathbf{t}_{21}]_\times\mathbf{R}_{21} = \underbrace{[\mathbf{R}_2\mathbf{b}]_\times}_{\text{baseline in Cam 2}}\mathbf{R}_{21} = \mathbf{R}_{21}\underbrace{[\mathbf{R}_1\mathbf{b}]_\times}_{\text{baseline in Cam 1}} = \mathbf{R}_{21}[-\mathbf{R}_{21}^\top\mathbf{t}_{21}]_\times$$ essential

►The Structure and the Key Properties of the Fundamental Matrix

$$\mathbf{F} = (\underbrace{\mathbf{Q}_2\mathbf{Q}_1^{-1}}_{\text{epipolar homography } \mathbf{H}_e})^{-\top}[\mathbf{e}_1]_\times = \underbrace{\mathbf{K}_2^{-\top}\mathbf{R}_{21}\mathbf{K}_1^{\top}}_{\mathbf{H}_e^{-\top}}\overbrace{[\mathbf{e}_1]_\times}^{\text{left epipole}} \overset{\to 76}{\simeq} \overbrace{[\mathbf{H}_e\mathbf{e}_1]_\times}^{\text{right epipole}}\mathbf{H}_e = \mathbf{K}_2^{-\top}\underbrace{[-\mathbf{t}_{21}]_\times\mathbf{R}_{21}}_{\text{essential matrix } \mathbf{E}}\mathbf{K}_1^{-1}$$

1. $\mathbf{E}$ captures relative camera pose only [Longuet-Higgins 1981]

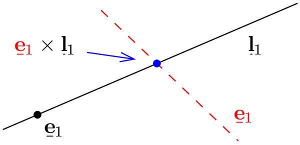
(the change of the world coordinate system does not change $\mathbf{E}$)

$$\begin{bmatrix}\mathbf{R}'_i & \mathbf{t}'_i\end{bmatrix} = \begin{bmatrix}\mathbf{R}_i & \mathbf{t}_i\end{bmatrix} \cdot \begin{bmatrix}\mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1\end{bmatrix} = \begin{bmatrix}\mathbf{R}_i\mathbf{R} & \mathbf{R}_i\mathbf{t} + \mathbf{t}_i\end{bmatrix},$$

then

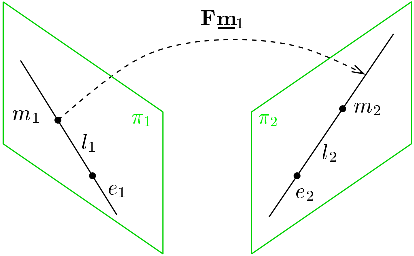
$$\mathbf{R}'_{21} = \mathbf{R}'_2\mathbf{R}'^{\top}_1 = \cdots = \mathbf{R}_{21} \qquad \mathbf{t}'_{21} = \mathbf{t}'_2 - \mathbf{R}'_{21}\mathbf{t}'_1 = \cdots = \mathbf{t}_{21}$$

2. the translation length $\mathbf{t}_{21}$ is lost since $\mathbf{E}$ is homogeneous
3. $\mathbf{F}$ maps points to lines and it is not a homography
4. $\mathbf{H}_e$ maps epipoles to epipoles, $\mathbf{H}_e^{-\top}$ epipolar lines to epipolar lines: $\underline{\mathbf{l}}_2 \simeq \mathbf{H}_e^{-\top}\underline{\mathbf{l}}_1$



- replacement for $\mathbf{H}_e^{-\top}$ for epipolar line map: $\underline{\mathbf{l}}_2 \simeq \mathbf{F}[\underline{\mathbf{e}}_1]_\times\underline{\mathbf{l}}_1$
- proof by point/line ‘transmutation’ (left)
- point $\underline{\mathbf{e}}_1$ does not lie on line $\underline{\mathbf{e}}_1$ (dashed): $\underline{\mathbf{e}}_1^\top\underline{\mathbf{e}}_1 \neq 0$
- $\mathbf{F}[\underline{\mathbf{e}}_1]_\times$ is not a homography, unlike $\mathbf{H}_e^{-\top}$ but it does the same job for epipolar line mapping

▶Summary: Relations and Mappings Involving Fundamental Matrix



$$0 = \underline{\mathbf{m}}_2^\top \mathbf{F}\, \underline{\mathbf{m}}_1$$

$$\begin{aligned}
\underline{\mathbf{e}}_1 &\simeq \operatorname{null}(\mathbf{F}), & \underline{\mathbf{e}}_2 &\simeq \operatorname{null}(\mathbf{F}^\top) \\
\underline{\mathbf{e}}_1 &\simeq \mathbf{H}_e^{-1}\underline{\mathbf{e}}_2 & \underline{\mathbf{e}}_2 &\simeq \mathbf{H}_e\underline{\mathbf{e}}_1 \\
\underline{\mathbf{l}}_1 &\simeq \mathbf{F}^\top\underline{\mathbf{m}}_2 & \underline{\mathbf{l}}_2 &\simeq \mathbf{F}\underline{\mathbf{m}}_1 \\
\underline{\mathbf{l}}_1 &\simeq \mathbf{H}_e^\top\underline{\mathbf{l}}_2 & \underline{\mathbf{l}}_2 &\simeq \mathbf{H}_e^{-\top}\underline{\mathbf{l}}_1 \\
\underline{\mathbf{l}}_1 &\simeq \mathbf{F}^\top[\underline{\mathbf{e}}_2]_\times\underline{\mathbf{l}}_2 & \underline{\mathbf{l}}_2 &\simeq \mathbf{F}[\underline{\mathbf{e}}_1]_\times\underline{\mathbf{l}}_1
\end{aligned}$$

$$\begin{tikzcd}
m_1 \arrow[d, "\mathbf{F}"'] & \underline{\mathbf{m}}_2^\top\mathbf{F}\underline{\mathbf{m}}_1 = 0 & m_2 \arrow[d, "\mathbf{F}^\top"] \\
l_2 \arrow[rr, bend right, "\mathbf{H}_e^\top \text{ or } \mathbf{F}^\top[\underline{\mathbf{e}}_2]_\times"'] & & l_1 \arrow[ll, bend right, "\mathbf{H}_e^{-\top} \text{ or } \mathbf{F}\,[\underline{\mathbf{e}}_1]_\times"']
\end{tikzcd}$$

- $\mathbf{F}[\underline{\mathbf{e}}_1]_\times$ maps lines to lines but it is not a homography
- $\mathbf{H}_e = \mathbf{Q}_2\mathbf{Q}_1^{-1}$ is the epipolar homography→78
  $\mathbf{H}_e^{-\top}$ maps epipolar lines to epipolar lines, where

$$\mathbf{H}_e = \mathbf{Q}_2\mathbf{Q}_1^{-1} = \mathbf{K}_2\mathbf{R}_{21}\mathbf{K}_1^{-1}$$

you have seen this →59

▶Representation Theorem for Fundamental Matrices

Def: $\mathbf{F}$ is fundamental when $\mathbf{F} \simeq \mathbf{H}^{-\top}[\underline{\mathbf{e}}_1]_\times$, where $\mathbf{H}$ is regular and $\underline{\mathbf{e}}_1 \simeq \operatorname{null} \mathbf{F} \neq \mathbf{0}$.

Theorem: A 3×3 matrix $\mathbf{A}$ is fundamental iff it is of rank 2.

Proof.
Direct: By the geometry, $\mathbf{H}$ is full-rank, $\underline{\mathbf{e}}_1 \neq \mathbf{0}$, hence $\mathbf{H}^{-\top}[\underline{\mathbf{e}}_1]_\times$ is a 3×3 matrix of rank 2.
Converse:

1. let $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$ be the SVD of $\mathbf{A}$ of rank 2; then $\mathbf{D} = \operatorname{diag}(\lambda_1, \lambda_2, 0)$, $\lambda_1 \geq \lambda_2 > 0$
2. we write $\mathbf{D} = \mathbf{B}\mathbf{C}$, where $\mathbf{B} = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$, $\mathbf{C} = \operatorname{diag}(1, 1, 0)$, $\lambda_3 = \lambda_2$ (w.l.o.g.)
3. then $\mathbf{A} = \mathbf{U}\mathbf{B}\mathbf{C}\mathbf{V}^\top = \mathbf{U}\mathbf{B}\mathbf{C}\underbrace{\mathbf{W}\mathbf{W}^\top}_{\mathbf{I}}\mathbf{V}^\top$ with $\mathbf{W}$ rotation
4. we look for a rotation $\mathbf{W}$ that maps $\mathbf{C}$ to a skew-symmetric $\mathbf{S}$, i.e. $\mathbf{S} = \mathbf{C}\mathbf{W}$
5. then $\mathbf{W} = \begin{bmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $|\alpha| = 1$, and $\mathbf{S} = [\mathbf{s}]_\times$, $\mathbf{s} = (0, 0, 1)$
6. we write $\mathbf{v}_3$ – 3rd column of $\mathbf{V}$, $\mathbf{u}_3$ – 3rd column of $\mathbf{U}$
$$\mathbf{A} = \mathbf{U}\mathbf{B}[\mathbf{s}]_\times \mathbf{W}^\top \mathbf{V}^\top \overset{\circledast 1}{=} \cdots = \underbrace{\mathbf{U}\mathbf{B}(\mathbf{V}\mathbf{W})^\top}_{\simeq \mathbf{H}^{-\top}} [\mathbf{v}_3]_\times \simeq \underbrace{[\mathbf{H}\mathbf{v}_3]_\times}_{\simeq [\mathbf{u}_3]_\times} \mathbf{H}, \tag{12}$$
7. $\mathbf{H}$ regular, $\mathbf{A}\mathbf{v}_3 = \mathbf{0}$, $\mathbf{u}_3\mathbf{A} = \mathbf{0}$ for $\mathbf{v}_3 \neq \mathbf{0}$, $\mathbf{u}_3 \neq \mathbf{0}$ □

- we also got a (non-unique: $\alpha = \pm 1$) decomposition formula for fundamental matrices
- it follows there is no constraint on $\mathbf{F}$ except the rank

▶Representation Theorem for Essential Matrices

Theorem

Let $\mathbf{E}$ *be a* 3×3 *matrix with SVD* $\mathbf{E} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$. *Then* $\mathbf{E}$ *is essential iff* $\mathbf{D} \simeq \mathrm{diag}(1,1,0)$.

Proof.

Direct:

If $\mathbf{E}$ is an essential matrix, then the epipolar homography matrix is a rotation matrix (→78), hence $\mathbf{H}^{-\top} \simeq \mathbf{U}\mathbf{B}(\mathbf{V}\mathbf{W})^\top$ in (12) must be (λ-scaled) orthogonal, therefore $\mathbf{B} = \lambda\mathbf{I}$.

Converse:

$\mathbf{E}$ is fundamental with $\mathbf{D} = \lambda\,\mathrm{diag}(1,1,0)$ then we do not need $\mathbf{B}$ (as if $\mathbf{B} = \lambda\mathbf{I}$) in (12) and $\mathbf{U}(\mathbf{V}\mathbf{W})^\top$ is orthogonal, as required.

□

▸Essential Matrix Decomposition

We are decomposing $\mathbf{E}$ to $\mathbf{E} \simeq [-\mathbf{t}_{21}]_\times \mathbf{R}_{21} = \mathbf{R}_{21}[-\mathbf{R}_{21}^\top \mathbf{t}_{21}]_\times$ [H&Z, sec. 9.6]

1. compute SVD of $\mathbf{E} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$ and verify $\mathbf{D} = \lambda\,\mathrm{diag}(1,1,0)$
2. ensure $\mathbf{U}$, $\mathbf{V}$ are rotation matrices by $\mathbf{U} \mapsto \det(\mathbf{U})\mathbf{U}$, $\mathbf{V} \mapsto \det(\mathbf{V})\mathbf{V}$
3. compute

$$\mathbf{R}_{21} = \mathbf{U} \underbrace{\begin{bmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{W}} \mathbf{V}^\top, \quad \mathbf{t}_{21} = -\beta\,\mathbf{u}_3, \qquad |\alpha| = 1, \quad \beta \neq 0 \tag{13}$$

Notes

- $\mathbf{v}_3 \simeq \mathbf{R}_{21}^\top \mathbf{t}_{21}$ by (12), hence $\mathbf{R}_{21}\mathbf{v}_3 \simeq \mathbf{t}_{21} \simeq \mathbf{u}_3$ since it must fall in left null space by $\mathbf{E} \simeq [\mathbf{u}_3]_\times \mathbf{R}_{21}$
- $\mathbf{t}_{21}$ is recoverable up to scale β and direction $\mathrm{sign}\,\beta$
- the result for $\mathbf{R}_{21}$ is unique up to $\alpha = \pm 1$ despite non-uniqueness of SVD
- the change of sign in α rotates the solution by 180° about $\mathbf{t}_{21}$

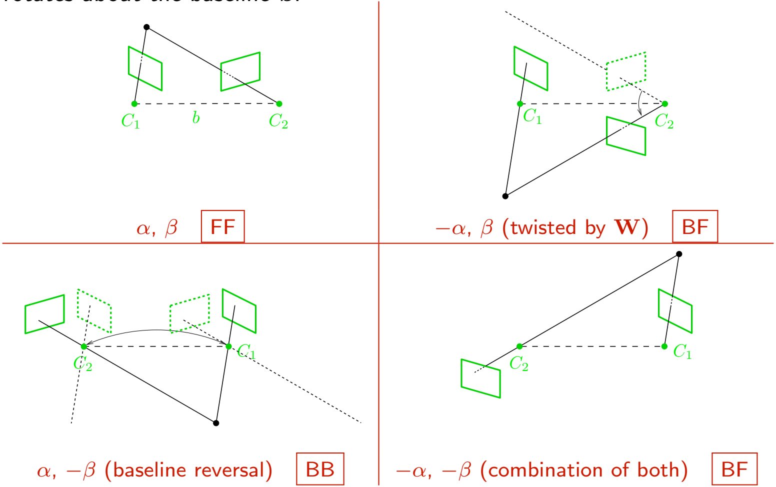
  $\mathbf{R}(\alpha) = \mathbf{U}\mathbf{W}\mathbf{V}^\top$, $\mathbf{R}(-\alpha) = \mathbf{U}\mathbf{W}^\top\mathbf{V}^\top \Rightarrow \mathbf{T} = \mathbf{R}(-\alpha)\mathbf{R}^\top(\alpha) = \cdots = \mathbf{U}\,\mathrm{diag}(-1,-1,1)\mathbf{U}^\top$
  which is a rotation by 180° about $\mathbf{u}_3 \simeq \mathbf{t}_{21}$: show that $\mathbf{u}_3$ is the rotation axis

$$\mathbf{U}\,\mathrm{diag}(-1,-1,1)\mathbf{U}^\top \mathbf{u}_3 = \mathbf{U}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{u}_3$$

- 4 solution sets for 4 sign combinations of α, β see next for geometric interpretation

▶Four Solutions to Essential Matrix Decomposition

Transform the world coordinate system so that the origin is in Camera 2. Then $\mathbf{t}_{21} = -\mathbf{b}$ and $\mathbf{W}$ rotates about the baseline $\mathbf{b}$. →77



- chirality constraint: all 3D points are in front of both cameras
- this singles-out the upper left case

[H&Z, Sec. 9.6.3]

▶We Have Added to The ZOO

continuation from →69

problem	given	unknown	slide
camera resection	6 world–img correspondences $\left\{(X_i, m_i)\right\}_{i=1}^{6}$	$\mathbf{P}$	62
exterior orientation	$\mathbf{K}$, 3 world–img correspondences $\left\{(X_i, m_i)\right\}_{i=1}^{3}$	$\mathbf{R}$, $\mathbf{t}$	66
relative orientation	3 world-world correspondences $\left\{(X_i, Y_i)\right\}_{i=1}^{3}$	$\mathbf{R}$, $\mathbf{t}$	70
fundamental matrix	7 img–img correspondences $\left\{(m_i, m_i')\right\}_{i=1}^{7}$	$\mathbf{F}$	84
relative orientation	$\mathbf{K}$, 5 img–img correspondences $\left\{(m_i, m_i')\right\}_{i=1}^{5}$	$\mathbf{R}$, $\mathbf{t}$	88
triangulation	$\mathbf{P}_1$, $\mathbf{P}_2$, 1 img–img correspondence (m_i, m_i')	X	89

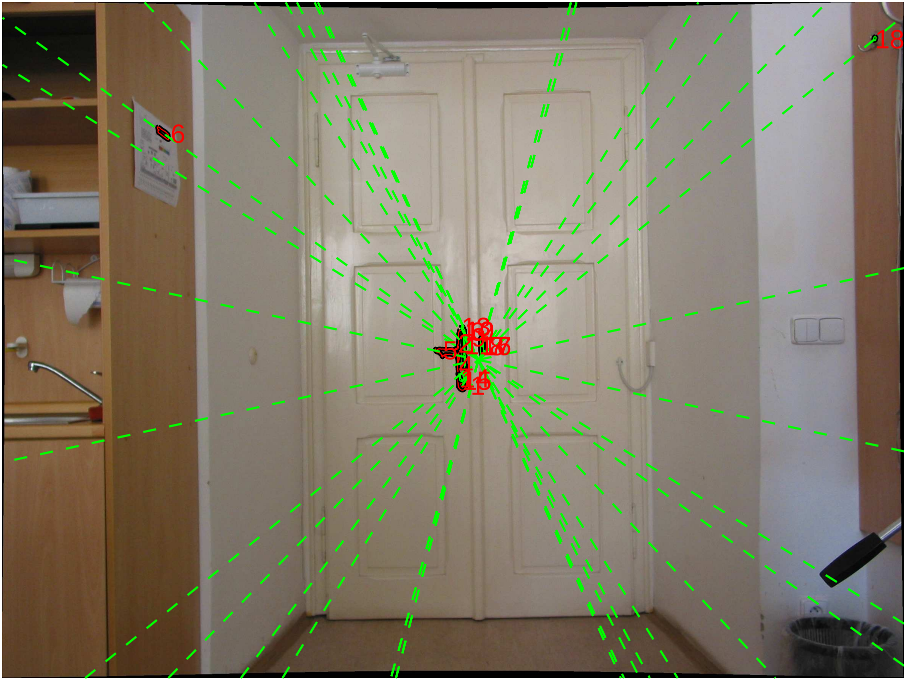
A bigger ZOO at http://cmp.felk.cvut.cz/minimal/

calibrated problems

- have fewer degenerate configurations
- can do with fewer points (good for geometry proposal generators →117)

- algebraic error optimization (SVD) makes sense in camera resection and triangulation only
- but it is not the best method; we will now focus on 'optimizing optimally'

Thank You


6
18