3D Computer Vision

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Open Informatics Master's Course

The Relative Orientation Problem

Problem: Given point triples (X_1, X_2, X_3) and (Y_1, Y_2, Y_3) in a general position in $\mathbf{R}^3$ such that the correspondence $X_i \leftrightarrow Y_i$ is known, determine the relative orientation $(\mathbf{R}, \mathbf{t})$ that maps $\mathbf{X}_i$ to $\mathbf{Y}_i$, i.e.

$$\underline{\mathbf{Y}_i = \mathbf{R}\mathbf{X}_i + \mathbf{t},} \quad i = 1, 2, 3\,. \qquad R^\top R = I$$

Applies to:

- 3D scanners
- partial reconstructions from different viewpoints

Obs: Let the centroid be $\bar{\mathbf{X}} = \frac{1}{3}\sum_i \mathbf{X}_i$ and analogically for $\bar{\mathbf{Y}}$. Then

$$\underline{\bar{\mathbf{Y}} = \mathbf{R}\bar{\mathbf{X}} + \mathbf{t}.}$$

Therefore

$$\mathbf{Z}_i \stackrel{\text{def}}{=} (\mathbf{Y}_i - \bar{\mathbf{Y}}) = \mathbf{R}(\mathbf{X}_i - \bar{\mathbf{X}}) \stackrel{\text{def}}{=} \mathbf{R}\mathbf{W}_i \qquad Z_i = R W_i$$

If all dot products are equal, $\mathbf{Z}_i^\top \mathbf{Z}_j = \mathbf{W}_i^\top \mathbf{W}_j$ for $i, j = 1, 2, 3$, we have $\quad Z_i W_i^{-1} = R$

$$\mathbf{R}^* = \begin{bmatrix} \mathbf{W}_1 & \mathbf{W}_2 & \mathbf{W}_3 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{Z}_1 & \mathbf{Z}_2 & \mathbf{Z}_3 \end{bmatrix} \qquad \text{NO WAY}$$

Otherwise (in practice) we setup a minimization problem $\quad \|a\|^2 = a^\top a$

$$\mathbf{R}^* = \arg\min_{\mathbf{R}} \sum_{i=1}^{3} \|\mathbf{Z}_i - \mathbf{R}\mathbf{W}_i\|^2 \quad \text{s.t.} \quad \mathbf{R}^\top\mathbf{R} = \mathbf{I}, \quad \det \mathbf{R} = 1$$

$$\arg\min_{\mathbf{R}} \sum_i \underline{\|\mathbf{Z}_i - \mathbf{R}\mathbf{W}_i\|^2} = \arg\min_{\mathbf{R}} \sum_i \left(\|\mathbf{Z}_i\|^2 - 2\mathbf{Z}_i^\top \mathbf{R}\mathbf{W}_i + \|\mathbf{W}_i\|^2 \right) = \cdots$$

$$\cdots = \arg\max_{\mathbf{R}} \sum_{i=1}^{3} \mathbf{Z}_i^\top \mathbf{R}\mathbf{W}_i$$

cont'd (What is Linear Algebra Telling Us?)

Obs 1: Let $\mathbf{A} : \mathbf{B} = \underbrace{\sum_{i,j} a_{ij} b_{ij}}$ be the dot-product (Frobenius inner product) over real matrices. Then

$$\mathbf{A} : \mathbf{B} = \mathbf{B} : \mathbf{A} = \operatorname{tr}(\mathbf{A}^\top \mathbf{B})$$

$\mathrm{vec}(A)^T \cdot \mathrm{vec}(B) = A : B$

Obs 2: (cyclic property for matrix trace)

$$\operatorname{tr}(\mathbf{ABC}) = \operatorname{tr}(\mathbf{CAB}) = \operatorname{tr}(BCA)$$

$c = \operatorname{tr}(c) \quad c \in R^1$

Obs 3: ($\mathbf{Z}_i$, $\mathbf{W}_i$ are vectors)

$$\underbrace{\mathbf{Z}_i^\top \mathbf{R} \mathbf{W}_i} = \operatorname{tr}(\mathbf{Z}_i^\top \mathbf{R} \mathbf{W}_i) = \operatorname{tr}(\mathbf{W}_i \mathbf{Z}_i^\top \mathbf{R}) = \Sigma\,(\mathbf{Z}_i \mathbf{W}_i^\top) : \mathbf{R} = \mathbf{R} : (\mathbf{Z}_i \mathbf{W}_i^\top)$$

Let the SVD be $\in R$

$$\sum_i \mathbf{Z}_i \mathbf{W}_i^\top \overset{\text{def}}{=} \mathbf{M} = \mathbf{UDV}^\top$$

Then

$$\mathbf{R} : \mathbf{M} = \mathbf{R} : (\mathbf{UDV}^\top) = \operatorname{tr}(\mathbf{R}^\top \mathbf{UDV}^\top) = \operatorname{tr}(\mathbf{V}^\top \mathbf{R}^\top \mathbf{UD}) = \underbrace{(\overbrace{\mathbf{U}^\top \mathbf{R} \mathbf{V}}^{\text{ortho}}) : \mathbf{D}}$$

cont'd: The Algorithm

We are solving

$$\mathbf{R}^* = \arg\max_{\mathbf{R}} \sum_i \mathbf{Z}_i^\top \mathbf{R} \mathbf{W}_i = \arg\max_{\mathbf{R}} \left(\mathbf{U}^\top \mathbf{R} \mathbf{V}\right) : \mathbf{D}$$

I

A particular solution is found as follows:

- $\mathbf{U}^\top \mathbf{R} \mathbf{V}$ must be (1) orthogonal, and most similar to (2) diagonal, (3) positive definite
- Since $\mathbf{U}$, $\mathbf{V}$ are orthogonal matrices then the solution to the problem is among $\mathbf{R}^* = \mathbf{U}\mathbf{S}\mathbf{V}^\top$, where $\mathbf{S}$ is diagonal and orthogonal, i.e. one of

$$\pm\,\mathrm{diag}(1,1,1), \quad \pm\,\mathrm{diag}(1,-1,-1), \quad \pm\,\mathrm{diag}(-1,1,-1), \quad \pm\,\mathrm{diag}(-1,-1,1)$$

- $\mathbf{U}^\top \mathbf{V}$ is not necessarily positive definite
- We choose $\mathbf{S}$ so that $(\mathbf{R}^*)^\top \mathbf{R}^* = \mathbf{I}$

Alg:

1. Compute matrix $\mathbf{M} = \sum_i \mathbf{Z}_i \mathbf{W}_i^\top$.
2. Compute SVD $\mathbf{M} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$.
3. Compute all $\mathbf{R}_k = \mathbf{U}\mathbf{S}_k\mathbf{V}^\top$ that give $\mathbf{R}_k^\top \mathbf{R}_k = \mathbf{I}$.
4. Compute $\mathbf{t}_k = \bar{\mathbf{Y}} - \mathbf{R}_k \bar{\mathbf{X}}$.

- The algorithm can be used for more than 3 points
- Triple pairs can be pre-filtered based on motion invariants (lengths, angles)
- The P3P problem is very similar but not identical

Module IV

Computing with a Camera Pair

4.1 Camera Motions Inducing Epipolar Geometry

4.2 Estimating Fundamental Matrix from 7 Correspondences

4.3 Estimating Essential Matrix from 5 Correspondences

4.4 Triangulation: 3D Point Position from a Pair of Corresponding Points

covered by

[1] [H&Z] Secs: 9.1, 9.2, 9.6, 11.1, 11.2, 11.9, 12.2, 12.3, 12.5.1

[2] H. Li and R. Hartley. Five-point motion estimation made easy. In *Proc ICPR* 2006, pp. 630–633

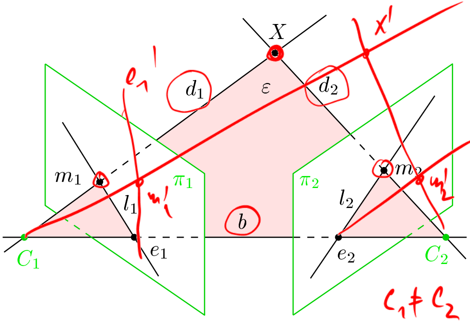
additional references

H. Longuet-Higgins. A computer algorithm for reconstructing a scene from two projections. *Nature*, 293 (5828):133–135, 1981.

▶Geometric Model of a Camera Pair

Epipolar geometry:

- brings constraints necessary for inter-image matching
- its parametric form encapsulates information about the relative pose of two cameras



two-camera setup

Description

- baseline b joins projection centers C_1, C_2

$$\mathbf{b} = \mathbf{C}_2 - \mathbf{C}_1$$

- epipole $e_i \in \pi_i$ is the image of C_j:

$$\underline{\mathbf{e}}_1 \simeq \mathbf{P}_1 \underline{\mathbf{C}}_2, \quad \underline{\mathbf{e}}_2 \simeq \mathbf{P}_2 \underline{\mathbf{C}}_1$$

- $l_i \in \pi_i$ is the image of epipolar plane

$$\varepsilon = (C_2, X, C_1)$$

- l_j is the epipolar line in image π_j induced by m_i in image π_i

Epipolar constraint: corresponding d_2, b, d_1 are coplanar

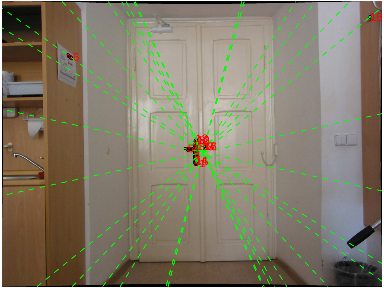
a necessary condition →87

$$\mathbf{P}_i = \begin{bmatrix} \mathbf{Q}_i & \mathbf{q}_i \end{bmatrix} = \mathbf{K}_i \begin{bmatrix} \mathbf{R}_i & \mathbf{t}_i \end{bmatrix} = \mathbf{K}_i \mathbf{R}_i \begin{bmatrix} \mathbf{I} & -\mathbf{C}_i \end{bmatrix} \qquad i = 1, 2$$

→31

Epipolar Geometry Example: Forward Motion

6 18



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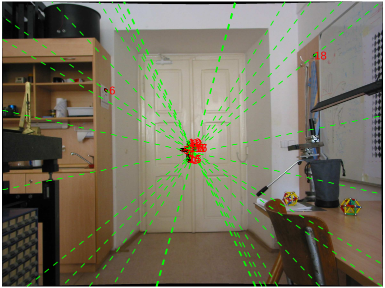


image 1

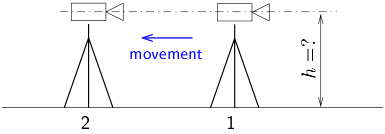
image 2

- red: correspondences
- green: epipolar line pairs per correspondence

click on the image to see their IDs

same ID in both images

How high was the camera above the floor?



▶Cross Products and Maps by Skew-Symmetric 3×3 Matrices

$\mathbf{b}, \mathbf{m} \in \mathbb{R}^3$ $\quad \in \mathbb{R}^{3\times 3}$

- There is an equivalence $\mathbf{b} \times \mathbf{m} = [\mathbf{b}]_\times \mathbf{m}$, where $[\mathbf{b}]_\times$ is a 3×3 skew-symmetric matrix

$$[\mathbf{b}]_\times = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}, \qquad \text{assuming} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

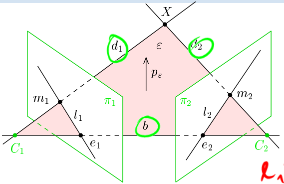
$[b]_\times^\top = -[b]_\times$

Some properties

1. $[\mathbf{b}]_\times^\top = -[\mathbf{b}]_\times$ — the general antisymmetry property
2. $\mathbf{A}$ is skew-symmetric iff $\mathbf{x}^\top \mathbf{A} \mathbf{x} = 0$ for all $\mathbf{x}$ — skew-sym mtx generalizes cross products
3. $[\mathbf{b}]_\times^3 = -\|\mathbf{b}\|^2 \cdot [\mathbf{b}]_\times$
4. $\|[\mathbf{b}]_\times\|_F = \sqrt{2}\,\|\mathbf{b}\|$ — Frobenius norm ($\|\mathbf{A}\|_F = \sqrt{\mathrm{tr}(\mathbf{A}^\top \mathbf{A})} = \sqrt{\sum_{i,j} |a_{ij}|^2}$)
5. $[\mathbf{b}]_\times \mathbf{b} = \mathbf{0}$ $\qquad b \times b = 0 \qquad [b]_\times b = 0$
6. $\mathrm{rank}\,[\mathbf{b}]_\times = 2$ iff $\|\mathbf{b}\| > 0$ — check minors of $[\mathbf{b}]_\times$
7. eigenvalues of $[\mathbf{b}]_\times$ are $(0, \lambda, -\lambda)$
8. for any 3×3 regular $\mathbf{B}$: $\quad \mathbf{B}^\top [\mathbf{B}\mathbf{z}]_\times \mathbf{B} = \det \mathbf{B}\, [\mathbf{z}]_\times$ — follows from the factoring on →39
9. in particular: if $\mathbf{R}\mathbf{R}^\top = \mathbf{I}$ then $\boxed{[\mathbf{R}\mathbf{b}]_\times = \mathbf{R}[\mathbf{b}]_\times \mathbf{R}^\top}$

- note that if $\mathbf{R}_b$ is rotation about $\mathbf{b}$ then $\mathbf{R}_b \mathbf{b} = \mathbf{b}$
- note $[\mathbf{b}]_\times$ is not a homography; it is not a rotation matrix — it is the logarithm of a rotation mtx

►Expressing Epipolar Constraint Algebraically



$$\mathbf{P}_i = \begin{bmatrix}\mathbf{Q}_i & \mathbf{q}_i\end{bmatrix} = \mathbf{K}_i \begin{bmatrix}\mathbf{R}_i & \mathbf{t}_i\end{bmatrix}, \quad i = 1, 2$$

$\mathbf{R}_{21}$ – relative camera rotation, $\mathbf{R}_{21} = \mathbf{R}_2 \mathbf{R}_1^\top$

$\mathbf{t}_{21}$ – relative camera translation, $\boxed{\mathbf{t}_{21} = \mathbf{t}_2 - \mathbf{R}_{21}\mathbf{t}_1 = -\mathbf{R}_2 \mathbf{b}}$ →74

$\mathbf{b}$ – baseline vector (world coordinate system)

$l_1 = e_1 \times m_1$

remember: $\mathbf{C} = -\mathbf{Q}^{-1}\mathbf{q} = -\mathbf{R}^\top \mathbf{t}$ →33 and 35

$$0 = \underbrace{\mathbf{d}_2^\top \mathbf{p}_\varepsilon}_{\text{normal of } \varepsilon} \simeq \underbrace{(\mathbf{Q}_2^{-1}\underline{\mathbf{m}}_2)^\top}_{\text{optical ray}} \underbrace{\mathbf{Q}_1^\top \underline{\mathbf{l}}_1}_{\text{optical plane}} = \underline{\mathbf{m}}_2^\top \underbrace{\mathbf{Q}_2^{-\top}\mathbf{Q}_1^\top (\underline{\mathbf{e}}_1 \times \underline{\mathbf{m}}_1)}_{\text{image of } \varepsilon \text{ in } \pi_2} = \underline{\mathbf{m}}_2^\top \underbrace{\left(\mathbf{Q}_2^{-\top}\mathbf{Q}_1^\top [\underline{\mathbf{e}}_1]_\times\right)}_{\text{fundamental matrix } \mathbf{F}} \underline{\mathbf{m}}_1$$

Epipolar constraint $\quad \underline{\mathbf{m}}_2^\top \mathbf{F}\, \underline{\mathbf{m}}_1 = 0 \quad$ is a point-line incidence constraint

l_2

- point $\underline{\mathbf{m}}_2$ is incident on epipolar line $\underline{\mathbf{l}}_2 \simeq \mathbf{F}\underline{\mathbf{m}}_1$
- point $\underline{\mathbf{m}}_1$ is incident on epipolar line $\underline{\mathbf{l}}_1 \simeq \mathbf{F}^\top \underline{\mathbf{m}}_2$
- $\mathbf{F}\underline{\mathbf{e}}_1 = \mathbf{F}^\top \underline{\mathbf{e}}_2 = \mathbf{0}$ (non-trivially)
- all epipolars meet at the epipole

$= -Q_1 C_1$

$$\underline{\mathbf{e}}_1 \simeq \mathbf{Q}_1 \mathbf{C}_2 + \mathbf{q}_1 = \mathbf{Q}_1(\mathbf{C}_2 - \mathbf{C}_1) = \mathbf{K}_1 \mathbf{R}_1 \mathbf{b} = -\mathbf{K}_1 \mathbf{R}_1 \mathbf{R}_2^\top \mathbf{t}_{21} = -\mathbf{K}_1 \mathbf{R}_{21}^\top \mathbf{t}_{21}$$

$$\mathbf{F} = \mathbf{Q}_2^{-\top}\mathbf{Q}_1^\top [\underline{\mathbf{e}}_1]_\times = \mathbf{Q}_2^{-\top}\mathbf{Q}_1^\top [-\mathbf{K}_1 \mathbf{R}_{21}^\top \mathbf{t}_{21}]_\times = \overset{\circledast 1}{\cdots} \simeq \mathbf{K}_2^{-\top}[-\mathbf{t}_{21}]_\times \mathbf{R}_{21} \mathbf{K}_1^{-1} \quad \text{fundamental}$$

$$\mathbf{E} = [-\mathbf{t}_{21}]_\times \mathbf{R}_{21} = \underbrace{[\mathbf{R}_2 \mathbf{b}]_\times}_{\text{baseline in Cam 2}} \mathbf{R}_{21} = \mathbf{R}_{21} \underbrace{[\mathbf{R}_1 \mathbf{b}]_\times}_{\text{baseline in Cam 1}} = \mathbf{R}_{21}[-\mathbf{R}_{21}^\top \mathbf{t}_{21}]_\times \quad \text{essential}$$

▶The Structure and the Key Properties of the Fundamental Matrix

$m_2^\top H_e [e_1]_\times m_1 = 0$

$$\mathbf{F} = (\underbrace{\mathbf{Q}_2\mathbf{Q}_1^{-1}}_{\text{epipolar homography } \mathbf{H}_e})^{-\top}[\mathbf{e}_1]_\times = \underbrace{\mathbf{K}_2^{-\top}\mathbf{R}_{21}\mathbf{K}_1^{\top}}_{\mathbf{H}_e^{-\top}}\,\overbrace{[\mathbf{e}_1]_\times}^{\text{left epipole}} \overset{\to 76}{\simeq} [\overbrace{\underbrace{\mathbf{H}_e\mathbf{e}_1}_{e_2}}^{\text{right epipole}}]_\times \mathbf{H}_e = \mathbf{K}_2^{-\top}\,\underbrace{[-\mathbf{t}_{21}]_\times \mathbf{R}_{21}}_{\text{essential matrix } \mathbf{E}}\,\mathbf{K}_1^{-1}$$

1. $\mathbf{E}$ captures relative camera pose only [Longuet-Higgins 1981]

(the change of the world coordinate system does not change $\mathbf{E}$)

$$\begin{bmatrix}\mathbf{R}'_i & \mathbf{t}'_i\end{bmatrix} = \begin{bmatrix}\mathbf{R}_i & \mathbf{t}_i\end{bmatrix} \cdot \begin{bmatrix}\mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1\end{bmatrix} = \begin{bmatrix}\mathbf{R}_i\mathbf{R} & \mathbf{R}_i\mathbf{t} + \mathbf{t}_i\end{bmatrix},$$

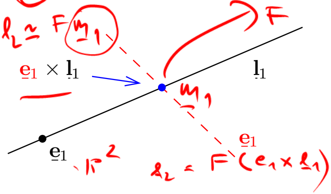
then

$$\mathbf{R}'_{21} = \mathbf{R}'_2\mathbf{R}'^{\top}_1 = \cdots = \mathbf{R}_{21} \qquad \mathbf{t}'_{21} = \mathbf{t}'_2 - \mathbf{R}'_{21}\mathbf{t}'_1 = \cdots = \mathbf{t}_{21}$$

2. the translation length $\mathbf{t}_{21}$ is lost since $\mathbf{E}$ is homogeneous
3. $\mathbf{F}$ maps points to lines and it is not a homography
4. $\mathbf{H}_e$ maps epipoles to epipoles, $\mathbf{H}_e^{-\top}$ epipolar lines to epipolar lines: $\underline{\mathbf{l}}_2 \simeq \mathbf{H}_e^{-\top}\underline{\mathbf{l}}_1$

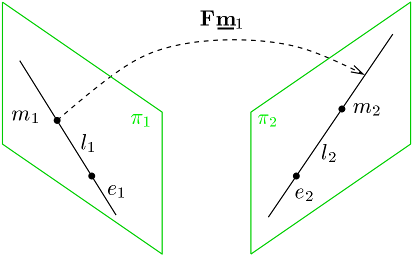
$m_2^\top F m_1 = 0$

$\lambda F \quad \lambda \neq 0$



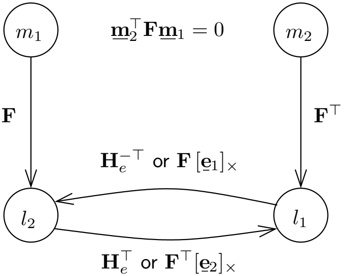
- replacement for $\mathbf{H}_e^{-\top}$ for epipolar line map: $\boxed{\underline{\mathbf{l}}_2 \simeq \mathbf{F}[\underline{\mathbf{e}}_1]_\times \underline{\mathbf{l}}_1}$
- proof by point/line 'transmutation' (left)
- point $\mathbf{e}_1$ does not lie on line $\underline{\mathbf{e}}_1$ (dashed): $\mathbf{e}_1^\top \underline{\mathbf{e}}_1 \neq 0$
- $\mathbf{F}[\underline{\mathbf{e}}_1]_\times$ is not a homography, unlike $\mathbf{H}_e^{-\top}$ but it does the same job for epipolar line mapping

▶Summary: Relations and Mappings Involving Fundamental Matrix



$$0 = \underline{\mathbf{m}}_2^\top \mathbf{F}\, \underline{\mathbf{m}}_1$$

$$\begin{array}{ll}
\underline{\mathbf{e}}_1 \simeq \mathrm{null}(\mathbf{F}), & \underline{\mathbf{e}}_2 \simeq \mathrm{null}(\mathbf{F}^\top) \\
\underline{\mathbf{e}}_1 \simeq \mathbf{H}_e^{-1} \underline{\mathbf{e}}_2 & \underline{\mathbf{e}}_2 \simeq \mathbf{H}_e \underline{\mathbf{e}}_1 \\
\underline{\mathbf{l}}_1 \simeq \mathbf{F}^\top \underline{\mathbf{m}}_2 & \underline{\mathbf{l}}_2 \simeq \mathbf{F} \underline{\mathbf{m}}_1 \\
\underline{\mathbf{l}}_1 \simeq \mathbf{H}_e^\top \underline{\mathbf{l}}_2 & \underline{\mathbf{l}}_2 \simeq \mathbf{H}_e^{-\top} \underline{\mathbf{l}}_1 \\
\underline{\mathbf{l}}_1 \simeq \mathbf{F}^\top [\underline{\mathbf{e}}_2]_\times \underline{\mathbf{l}}_2 & \underline{\mathbf{l}}_2 \simeq \mathbf{F} [\underline{\mathbf{e}}_1]_\times \underline{\mathbf{l}}_1
\end{array}$$



- $\mathbf{F}[\underline{\mathbf{e}}_1]_\times$ maps lines to lines but it is not a homography
- $\mathbf{H}_e = \mathbf{Q}_2 \mathbf{Q}_1^{-1}$ is the epipolar homography→78
  $\mathbf{H}_e^{-\top}$ maps epipolar lines to epipolar lines, where

$$\mathbf{H}_e = \mathbf{Q}_2 \mathbf{Q}_1^{-1} = \mathbf{K}_2 \mathbf{R}_{21} \mathbf{K}_1^{-1}$$

you have seen this →59

▶Representation Theorem for Fundamental Matrices

Def: $\mathbf{F}$ is fundamental when $\mathbf{F} \simeq \mathbf{H}^{-\top}[\underline{\mathbf{e}}_1]_\times$ where $\mathbf{H}$ is regular and $\underline{\mathbf{e}}_1 \simeq \operatorname{null} \mathbf{F} \neq \mathbf{0}$.

Theorem: A 3×3 matrix $\mathbf{A}$ is fundamental iff it is of rank 2.

Proof.

Direct: By the geometry, $\mathbf{H}$ is full-rank, $\underline{\mathbf{e}}_1 \neq \mathbf{0}$, hence $\mathbf{H}^{-\top}[\underline{\mathbf{e}}_1]_\times$ is a 3×3 matrix of rank 2.

Converse:

1. let $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$ be the SVD of $\mathbf{A}$ of rank 2; then $\mathbf{D} = \operatorname{diag}(\lambda_1, \lambda_2, 0)$, $\lambda_1 \geq \lambda_2 > 0$
2. we write $\mathbf{D} = \mathbf{B}\mathbf{C}$, where $\mathbf{B} = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$, $\mathbf{C} = \operatorname{diag}(1, 1, 0)$, $\lambda_3 = \lambda_2$ (w.l.o.g.)
3. then $\mathbf{A} = \mathbf{U}\mathbf{B}\mathbf{C}\mathbf{V}^\top = \mathbf{U}\mathbf{B}\mathbf{C}\underbrace{\mathbf{W}\mathbf{W}^\top}_{\mathbf{I}}\mathbf{V}^\top$ with $\mathbf{W}$ rotation
4. we look for a rotation $\mathbf{W}$ that maps $\mathbf{C}$ to a skew-symmetric $\mathbf{S}$, i.e. $\mathbf{S} = \mathbf{C}\mathbf{W}$
5. then $\mathbf{W} = \begin{bmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $|\alpha| = 1$, and $\mathbf{S} = [\mathbf{s}]_\times$, $\mathbf{s} = (0, 0, 1)$
6. we write

   $\mathbf{v}_3$ – 3rd column of $\mathbf{V}$, $\mathbf{u}_3$ – 3rd column of $\mathbf{U}$

$$\mathbf{A} = \mathbf{U}\mathbf{B}[\mathbf{s}]_\times \mathbf{W}^\top \mathbf{V}^\top = \overset{\circledast 1}{\cdots} = \underbrace{\mathbf{U}\mathbf{B}(\mathbf{V}\mathbf{W})^\top}_{\simeq \mathbf{H}^{-\top}}[\mathbf{v}_3]_\times \simeq \underbrace{[\mathbf{H}\mathbf{v}_3]_\times}_{\simeq [\mathbf{u}_3]_\times}\mathbf{H}, \tag{12}$$

7. $\mathbf{H}$ regular, $\mathbf{A}\mathbf{v}_3 = \mathbf{0}$, $\mathbf{u}_3\mathbf{A} = \mathbf{0}$ for $\mathbf{v}_3 \neq \mathbf{0}$, $\mathbf{u}_3 \neq \mathbf{0}$ □

- we also got a (non-unique: $\alpha = \pm 1$) decomposition formula for fundamental matrices
- it follows there is no constraint on $\mathbf{F}$ except the rank

▶Representation Theorem for Essential Matrices

Theorem

Let $\mathbf{E}$ be a 3×3 matrix with SVD $\mathbf{E} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$. Then $\mathbf{E}$ is essential iff $\mathbf{D} \simeq \mathrm{diag}(1, 1, 0)$.

Proof.

Direct:

If $\mathbf{E}$ is an essential matrix, then the epipolar homography matrix is a rotation matrix (→78), hence $\mathbf{H}^{-\top} \simeq \mathbf{U}\mathbf{B}(\mathbf{V}\mathbf{W})^\top$ in (12) must be (λ-scaled) orthogonal, therefore $\mathbf{B} = \lambda\mathbf{I}$.

Converse:

$\mathbf{E}$ is fundamental with $\mathbf{D} = \lambda\,\mathrm{diag}(1, 1, 0)$ then we do not need $\mathbf{B}$ (as if $\mathbf{B} = \lambda\mathbf{I}$) in (12) and $\mathbf{U}(\mathbf{V}\mathbf{W})^\top$ is orthogonal, as required.

□

▶Essential Matrix Decomposition

We are decomposing $\mathbf{E}$ to $\mathbf{E} \simeq [-\mathbf{t}_{21}]_\times \mathbf{R}_{21} = \mathbf{R}_{21}[-\mathbf{R}_{21}^\top \mathbf{t}_{21}]_\times$ [H&Z, sec. 9.6]

1. compute SVD of $\overset{(1)}{\mathbf{E}} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$ and verify $\mathbf{D} = \lambda\,\mathrm{diag}(1,1,0)$
2. ensure $\mathbf{U}$, $\mathbf{V}$ are rotation matrices by $\mathbf{U} \mapsto \det(\mathbf{U})\mathbf{U}$, $\mathbf{V} \mapsto \det(\mathbf{V})\mathbf{V}$
3. compute

$$\mathbf{R}_{21} = \mathbf{U} \underbrace{\begin{bmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{W}} \mathbf{V}^\top, \quad \mathbf{t}_{21} = -\beta\,\mathbf{u}_3, \qquad |\alpha| = 1, \quad \beta \neq 0 \tag{13}$$

Notes

- $\mathbf{v}_3 \simeq \mathbf{R}_{21}^\top \mathbf{t}_{21}$ by (12), hence $\mathbf{R}_{21}\mathbf{v}_3 \simeq \mathbf{t}_{21} \simeq \mathbf{u}_3$ since it must fall in left null space by $\mathbf{E} \simeq [\mathbf{u}_3]_\times \mathbf{R}_{21}$
- $\mathbf{t}_{21}$ is recoverable up to scale β and direction $\mathrm{sign}\,\beta$
- the result for $\mathbf{R}_{21}$ is unique up to $\alpha = \pm 1$ despite non-uniqueness of SVD
- the change of sign in α rotates the solution by 180° about $\mathbf{t}_{21}$

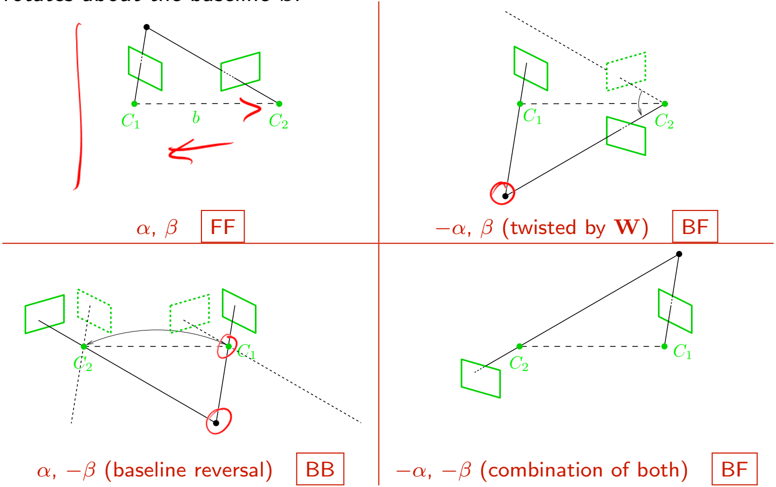
  $\mathbf{R}(\alpha) = \mathbf{U}\mathbf{W}\mathbf{V}^\top$, $\mathbf{R}(-\alpha) = \mathbf{U}\mathbf{W}^\top\mathbf{V}^\top \Rightarrow \mathbf{T} = \mathbf{R}(-\alpha)\mathbf{R}^\top(\alpha) = \cdots = \mathbf{U}\,\mathrm{diag}(-1,-1,1)\mathbf{U}^\top = T$
  which is a rotation by 180° about $\mathbf{u}_3 \simeq \mathbf{t}_{21}$: show that $\mathbf{u}_3$ is the rotation axis

$$\underbrace{\mathbf{U}\,\mathrm{diag}(-1,-1,1)\mathbf{U}^\top}_{T}\mathbf{u}_3 = \mathbf{U} \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{(0,0,1)} = \mathbf{u}_3$$

- 4 solution sets for 4 sign combinations of α, β see next for geometric interpretation

▶Four Solutions to Essential Matrix Decomposition

Transform the world coordinate system so that the origin is in Camera 2. Then $\mathbf{t}_{21} = -\mathbf{b}$ and $\mathbf{W}$ rotates about the baseline $\mathbf{b}$. →77



- chirality constraint: all 3D points are in front of both cameras
- this singles-out the upper left case

[H&Z, Sec. 9.6.3]

Thank You