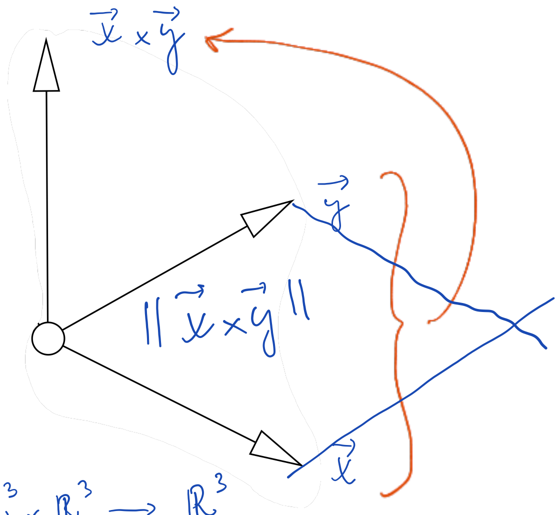
the notation introduced above, we are getting a version of the generalized Laplace expansion of the determinant [9, 10]

$$|\mathtt{A}| = \sum_{\omega\in\Omega} \left(\prod_{i\in[k],j\in[k+1,n]} \operatorname{sgn}(\varphi_{\overline{\omega}}(j) - \varphi_{\omega}(i)) \right) \left|\mathtt{A}^{\epsilon,\varphi_\omega}\right| \left|\mathtt{A}^{\rho,\varphi_{\overline{\omega}}(\rho)}\right| \tag{1.35}$$



1.3 Vector product (cross product ×)

$\times : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$$\underbrace{\Big(\underbrace{\big(\underbrace{(\mathbb{R}^3, \oplus)}_{\text{vectors}}, \underbrace{(\mathbb{R}, +, \cdot)}_{\text{scalars}}\big)}_{\text{Linear space}}, \odot\Big)}, \times \Big)$$

(Non-associative, non-commutative algebra)

Let us look at an interesting mapping from $\mathbb{R}^3 \times \mathbb{R}^3$ to $\mathbb{R}^3$, the *vector product* in $\mathbb{R}^3$ [4] (which it also often called the cross product [2]). Vector product has interesting geometrical properties but we shall motivate it by its connection to systems of linear equations.

§ 1 **Vector product** Assume two linearly independent coordinate vectors $\vec{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top$ and $\vec{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^\top$ in $\mathbb{R}^3$. The following system of linear equations

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \vec{z} = 0 \tag{1.36}$$

Cross (vector) product as a solution to a system of homogeneous linear equations

has a one-dimensional subspace V of solutions in $\mathbb{R}^3$. The solutions can be written as multiples of one non-zero vector $\vec{w}$, the basis of V, i.e.

$$\vec{z} = \lambda\,\vec{w}, \quad \lambda \in \mathbb{R} \tag{1.37}$$

Let us see how we can construct $\vec{w}$ in a convenient way from vectors $\vec{x}$, $\vec{y}$.

Consider determinants of two matrices constructed from the matrix of the system (1.36) by adjoining its first, resp. second, row to the matrix of the system (1.36)

3x3

$$\left|\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \end{bmatrix}\right| = 0 \qquad \left|\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ y_1 & y_2 & y_3 \end{bmatrix}\right| = 0 \tag{1.38}$$

rank = 2 < 3

which gives $\vec{x} = [x_1\ x_2\ y_3]$ $\quad \vec{y} = [y_1\ y_2\ y_3]$

2 determinants 3×3 $\rightarrow$ deg 3 in els of [ ]

$$x_1\,(x_2\,y_3 - x_3\,y_2) + x_2\,(x_3\,y_1 - x_1\,y_3) + x_3\,(x_1\,y_2 - x_2\,y_1) \;=\; 0 \tag{1.39}$$

$$y_1\,(x_2\,y_3 - x_3\,y_2) + y_2\,(x_3\,y_1 - x_1\,y_3) + y_3\,(x_1\,y_2 - x_2\,y_1) \;=\; 0 \tag{1.40}$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \vec{w} = 0$$

and can be rewritten as

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} x_2\,y_3 - x_3\,y_2 \\ -x_1\,y_3 + x_3\,y_1 \\ x_1\,y_2 - x_2\,y_1 \end{bmatrix} = 0 \tag{1.41}$$

We see that vector

$$\vec{w} = \begin{bmatrix} x_2\,y_3 - x_3\,y_2 \\ -x_1\,y_3 + x_3\,y_1 \\ x_1\,y_2 - x_2\,y_1 \end{bmatrix} \tag{1.42}$$

familiar formula

solves Equation 1.36.

Notice that elements of $\vec{w}$ are the three two by two minors of the matrix of the system (1.36). The rank of the matrix is two, which means that at least one of the minors is non-zero, and hence $\vec{w}$ is also non-zero. We see that $\vec{w}$ is a basic vector of V. Formula 1.42 is known as the *vector product* in $\mathbb{R}^3$ and $\vec{w}$ is also often denoted by $\vec{x} \times \vec{y}$.

$\times : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$

defined in terms of coordinates

$$\boxed{\vec{x}_{\beta'} = A\,\vec{x}_\beta}$$

§2 Vector product under the change of basis Let us next study the behavior of the vector product under the change of basis in $\mathbb{R}^3$. Let us have two bases β, β' in $\mathbb{R}^3$ and two vectors $\vec{x}$, $\vec{y}$ with coordinates $\vec{x}_\beta = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top$, $\vec{y}_\beta = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^\top$ and $\vec{x}_{\beta'} = \begin{bmatrix} x_1' & x_2' & x_3' \end{bmatrix}^\top$, $\vec{y}_\beta = \begin{bmatrix} y_1' & y_2' & y_3' \end{bmatrix}^\top$. We introduce

?

$$\underbrace{\vec{x}_\beta \times \vec{y}_\beta}_{\beta} = \begin{bmatrix} x_2\,y_3 - x_3\,y_2 \\ -x_1\,y_3 + x_3\,y_1 \\ x_1\,y_2 - x_2\,y_1 \end{bmatrix} \qquad \underbrace{\vec{x}_{\beta'} \times \vec{y}_{\beta'}}_{\beta'} = \begin{bmatrix} x_2'y_3' - x_3'y_2' \\ -x_1'y_3' + x_3'y_1' \\ x_1'y_2' - x_2'y_1' \end{bmatrix} \tag{1.43}$$

Technical derivation

To find the relationship between $\vec{x}_\beta \times \vec{y}_\beta$ and $\vec{x}_{\beta'} \times \vec{y}_{\beta'}$, we will use the following fact. For every three vectors $\vec{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top$, $\vec{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^\top$, $\vec{z} = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^\top$ in $\mathbb{R}^3$ there holds

$$\vec{z}^\top(\vec{x}\times\vec{y}) = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} x_2 y_3 - x_3 y_2 \\ -x_1 y_3 + x_3 y_1 \\ x_1 y_2 - x_2 y_1 \end{bmatrix} = \left| \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \right| = \left| \begin{bmatrix} \vec{x}^\top \\ \vec{y}^\top \\ \vec{z}^\top \end{bmatrix} \right| \quad (1.44)$$

$\vec{z}^{\,\top}(\vec{x}\times\vec{y}) \in \mathbb{R}$

$[\;.\;.\;.\;]$ $\quad \begin{bmatrix} \vdots \end{bmatrix}$ $\quad \left|\begin{bmatrix} \vec{x}^\top \\ \vec{y}^\top \\ \vec{z}^\top \end{bmatrix}\right|$

We can write

1. st coord

$$\begin{aligned}
\underbrace{\vec{x}_{\beta'} \times \vec{y}_{\beta'}}_{\vec{w}_{\beta'}} &= \begin{array}{c} 1 \\ 2 \\ 3 \end{array}\begin{bmatrix} [1\,0\,0]\,(\vec{x}_{\beta'} \times \vec{y}_{\beta'}) \\ [0\,1\,0]\,(\vec{x}_{\beta'} \times \vec{y}_{\beta'}) \\ [0\,0\,1]\,(\vec{x}_{\beta'} \times \vec{y}_{\beta'}) \end{bmatrix} = \begin{bmatrix} \left|\begin{bmatrix} \vec{x}_{\beta'}^\top \\ \vec{y}_{\beta'}^\top \\ 1\,0\,0 \end{bmatrix}\right| & \left|\begin{bmatrix} \vec{x}_{\beta'}^\top \\ \vec{y}_{\beta'}^\top \\ 0\,1\,0 \end{bmatrix}\right| & \left|\begin{bmatrix} \vec{x}_{\beta'}^\top \\ \vec{y}_{\beta'}^\top \\ 0\,0\,1 \end{bmatrix}\right| \end{bmatrix}^\top \\
&= \begin{bmatrix} \left|\begin{bmatrix} \vec{x}_\beta^\top \mathtt{A}^\top \\ \vec{y}_\beta^\top \mathtt{A}^\top \\ 1\,0\,0 \end{bmatrix}\right| & \left|\begin{bmatrix} \vec{x}_\beta^\top \mathtt{A}^\top \\ \vec{y}_\beta^\top \mathtt{A}^\top \\ 0\,1\,0 \end{bmatrix}\right| & \left|\begin{bmatrix} \vec{x}_\beta^\top \mathtt{A}^\top \\ \vec{y}_\beta^\top \mathtt{A}^\top \\ 0\,0\,1 \end{bmatrix}\right| \end{bmatrix}^\top \\
&= \begin{bmatrix} \left|\begin{bmatrix} \vec{x}_\beta^\top \\ \vec{y}_\beta^\top \\ [1\,0\,0]\,\mathtt{A}^{-\top} \end{bmatrix} \mathtt{A}^\top\right| & \left|\begin{bmatrix} \vec{x}_\beta^\top \\ \vec{y}_\beta^\top \\ [0\,1\,0]\,\mathtt{A}^{-\top} \end{bmatrix} \mathtt{A}^\top\right| & \left|\begin{bmatrix} \vec{x}_\beta^\top \\ \vec{y}_\beta^\top \\ [0\,0\,1]\,\mathtt{A}^{-\top} \end{bmatrix} \mathtt{A}^\top\right| \end{bmatrix}^\top \\
&= \begin{bmatrix} [1\,0\,0]\,\mathtt{A}^{-\top}(\vec{x}_\beta \times \vec{y}_\beta) \\ [0\,1\,0]\,\mathtt{A}^{-\top}(\vec{x}_\beta \times \vec{y}_\beta) \\ [0\,0\,1]\,\mathtt{A}^{-\top}(\vec{x}_\beta \times \vec{y}_\beta) \end{bmatrix} |\mathtt{A}^\top| \\
&= \frac{\mathtt{A}^{-\top}}{|\mathtt{A}^{-\top}|}\,\underbrace{(\vec{x}_\beta \times \vec{y}_\beta)}_{\vec{w}_\beta}
\end{aligned} \quad (1.45)$$

get it out

$\to \mathtt{I}$

$\vec{x}_{\beta'} = \mathtt{A}\vec{x}_\beta \Rightarrow \vec{x}_{\beta'}^{\,\top} = \vec{x}_\beta^{\,\top}\mathtt{A}^\top$

$\vec{w}_{\beta'} = \dfrac{\mathtt{A}^{-\top}}{|\mathtt{A}^{-\top}|}\,\vec{w}_\beta$

$|\mathtt{A}\,\mathtt{B}| = |\mathtt{A}|\,|\mathtt{B}|$

$|\mathtt{A}| = \dfrac{1}{|\mathtt{A}^{-1}|}$

2.

§ 3 Vector product as a linear mapping It is interesting to see that for all $\vec{x}, \vec{y} \in \mathbb{R}^3$ there holds

$$\vec{x} \times \vec{y} = \begin{bmatrix} x_2 y_3 - x_3 y_2 \\ -x_1 y_3 + x_3 y_1 \\ x_1 y_2 - x_2 y_1 \end{bmatrix} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad (1.46)$$

and thus we can introduce matrix

$$[\vec{x}]_\times = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \tag{1.47}$$

and write

$$\vec{x} \times \vec{y} = [\vec{x}]_\times \, \vec{y} \tag{1.48}$$

Notice also that $[\vec{x}]_\times^\top = -[\vec{x}]_\times$ and therefore

$$(\vec{x} \times \vec{y})^\top = ([\vec{x}]_\times \, \vec{y})^\top = -\vec{y}^\top \, [\vec{x}]_\times \tag{1.49}$$

CLEAR

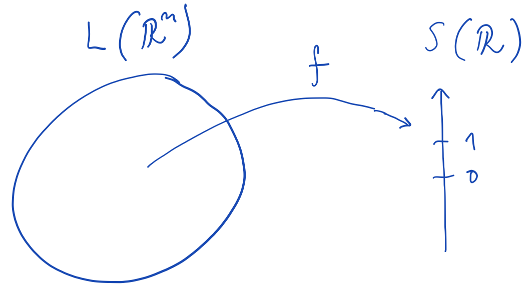
The result of § 2 can also be written in the formalism of this paragraph. We can write for every $\vec{x}, \vec{y} \in \mathbb{R}^3$

$$[\mathtt{A}\,\vec{x}_\beta]_\times \, \mathtt{A}\,\vec{y}_\beta = (\mathtt{A}\,\vec{x}_\beta) \times (\mathtt{A}\,\vec{y}_\beta) = \frac{\mathtt{A}^{-\top}}{|\mathtt{A}^{-\top}|}\,(\vec{x}_\beta \times \vec{y}_\beta) = \frac{\mathtt{A}^{-\top}}{|\mathtt{A}^{-\top}|}\,[\vec{x}_\beta]_\times \, \vec{y}_\beta \tag{1.50}$$

Linear functions on $L(\mathbb{R}^n, \mathbb{R}^3, \ldots)$

and hence we get for every $\vec{x} \in \mathbb{R}^3$

$$\boxed{[\mathtt{A}\,\vec{x}_\beta]_\times \, \mathtt{A} = \frac{\mathtt{A}^{-\top}}{|\mathtt{A}^{-\top}|}\,[\vec{x}_\beta]_\times} \tag{1.51}$$



1.4 Dual space and dual basis

Let us start with a three-dimensional linear space L over scalars S and consider the set $L^\star$ of all linear functions $f\colon L \to S$, i.e. the functions on L for which the following holds true

f ... linear function

affine

$$f(a\,\vec{x} + b\,\vec{y}) = a\,f(\vec{x}) + b\,f(\vec{y}) \tag{1.52}$$

for all $a, b \in S$ and all $\vec{x}, \vec{y} \in L$.

$y = \cancel{a x + b}$

NOT LINEAR HERE

Let us next define the addition $+^\star \colon L^\star \times L^\star \to L^\star$ of linear functions $f, g \in L^\star$ and the multiplication $\cdot^\star \colon S \times L^\star \to L^\star$ of a linear function $f \in L^\star$ by a scalar $a \in S$ such that

pointwise

Add:
$$(f +^\star g)(\vec{x}) = f(\vec{x}) + g(\vec{x}) \tag{1.53}$$
multiply:
$$(a \cdot^\star f)(\vec{x}) = a\, f(\vec{x}) \tag{1.54}$$

holds true for all $a \in S$ and for all $\vec{x} \in L$. One can verify that $(L^\star, +^\star, \cdot^\star)$ over $(S, +, \,)$ is itself a linear space [1, 4, 3]. It makes therefore a good sense to use arrows above symbols for linear functions, e.g. $\vec{f}$ instead of f.

The linear space $L^\star$ is derived from, and naturally connected to, the linear space L and hence deserves a special name. Linear space $L^\star$ is called [1] the *dual (linear) space* to L.

Now, consider a basis $\beta = [\vec{b}_1, \vec{b}_2, \vec{b}_3]$ of L. We will construct a basis $\beta^\star$ of $L^\star$, in a certain natural and useful way. Let us take three linear functions $\vec{b}_1^\star, \vec{b}_2^\star, \vec{b}_3^\star \in L^\star$ such that

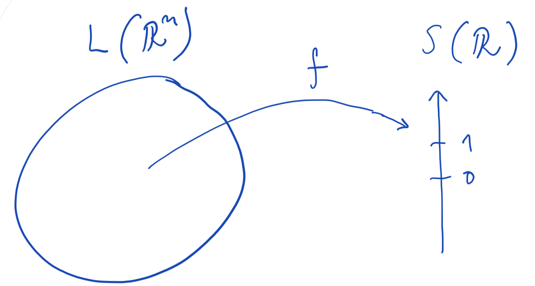
Definition relations

~ dim=3

$$\begin{array}{lll} \vec{b}_1^\star(\vec{b}_1) = 1 & \vec{b}_1^\star(\vec{b}_2) = 0 & \vec{b}_1^\star(\vec{b}_3) = 0 \\ \vec{b}_2^\star(\vec{b}_1) = 0 & \vec{b}_2^\star(\vec{b}_2) = 1 & \vec{b}_2^\star(\vec{b}_3) = 0 \\ \vec{b}_3^\star(\vec{b}_1) = 0 & \vec{b}_3^\star(\vec{b}_2) = 0 & \vec{b}_3^\star(\vec{b}_3) = 1 \end{array} \tag{1.55}$$

where 0 and 1 are the zero and the unit element of S, respectively. First of all, one has to verify [1] that such an assignment is possible with linear functions over L. Secondly one can show [1] that functions $\vec{b}_1^\star, \vec{b}_2^\star, \vec{b}_3^\star$ are determined by this assignment uniquely on all vectors of L. Finally, one can observe [1] that the triple $\beta^\star = [\vec{b}_1^\star, \vec{b}_2^\star, \vec{b}_3^\star]$ forms an (ordered) basis of $\vec{L}$. The basis $\beta^\star$ is called the *dual basis* of $L^\star$, i.e. it is the basis of $L^\star$, which is related in a special (dual) way to the basis β of L.

§ 1 Evaluating linear functions Consider a vector $\vec{x} \in L$ with coordinates $\vec{x}_\beta = [x_1, x_2, x_3]^\top$ w.r.t. a basis $\beta = [\vec{b}_1, \vec{b}_2, \vec{b}_3]$ and a linear function $\vec{h} \in$



$\vec{b}_i^* \equiv f$ ... linear function $\mathbb{R}^n \to \mathbb{R}$

$L^* = \{ f \mid f \equiv \text{linear function } L \to S \}$

|||

$\underbrace{((L^*, +^*), (S, +, \cdot), \cdot^*)}$ check axioms

Dual linear space to L :-)

L ($\mathbb{R}^3$) has basis β

L^* has basis β^*

related NICE

such that

$\vec{b}_i^*$ is a linear function on L

This is special basis $\boxed{\text{DUAL BASIS}}$

§1 Evaluating linear functions Consider a vector $\vec{x} \in L$ with coordinates $\vec{x}_\beta = [x_1, x_2, x_3]^\top$ w.r.t. a basis $\beta = [\vec{b}_1, \vec{b}_2, \vec{b}_3]$ and a linear function $\vec{h} \in$

$L^\star$ with coordinates $\vec{h}_{\beta^\star} = [h_1, h_2, h_3]^\top$ w.r.t. the dual basis $\beta^\star = [\vec{b}_1^\star, \vec{b}_2^\star, \vec{b}_3^\star]$. The value $\vec{h}(\vec{x}) \in S$ is obtained from the coordinates $\vec{x}_\beta$ and $\vec{h}_{\beta^\star}$ as

$$\vec{h}(\vec{x}) = \vec{h}(x_1\,\vec{b}_1 + x_2\,\vec{b}_2 + x_3\,\vec{b}_3) \tag{1.56}$$

$$= (h_1\,\vec{b}_1^\star + h_2\,\vec{b}_2^\star + h_3\,\vec{b}_3^\star)(x_1\,\vec{b}_1 + x_2\,\vec{b}_2 + x_3\,\vec{b}_3) \tag{1.57}$$

$$\begin{aligned} = \; & h_1\,\vec{b}_1^\star(\vec{b}_1)\,x_1 + h_1\,\vec{b}_1^\star(\vec{b}_2)\,x_2 + h_1\,\vec{b}_1^\star(\vec{b}_3)\,x_3 \\ & + h_2\,\vec{b}_2^\star(\vec{b}_1)\,x_1 + h_2\,\vec{b}_2^\star(\vec{b}_2)\,x_2 + h_2\,\vec{b}_2^\star(\vec{b}_3)\,x_3 \\ & + h_3\,\vec{b}_3^\star(\vec{b}_1)\,x_1 + h_3\,\vec{b}_3^\star(\vec{b}_2)\,x_2 + h_3\,\vec{b}_3^\star(\vec{b}_3)\,x_3 \end{aligned} \tag{1.58}$$

$$= \begin{bmatrix} h_1 & h_2 & h_3 \end{bmatrix} \begin{bmatrix} \vec{b}_1^\star(\vec{b}_1) & \vec{b}_1^\star(\vec{b}_2) & \vec{b}_1^\star(\vec{b}_3) \\ \vec{b}_2^\star(\vec{b}_1) & \vec{b}_2^\star(\vec{b}_2) & \vec{b}_2^\star(\vec{b}_3) \\ \vec{b}_3^\star(\vec{b}_1) & \vec{b}_3^\star(\vec{b}_2) & \vec{b}_3^\star(\vec{b}_3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{1.59}$$

$$= \begin{bmatrix} h_1 & h_2 & h_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{1.60}$$

$$= \begin{bmatrix} h_1, h_2, h_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{1.61}$$

$$= \vec{h}_{\beta^\star}^\top\, \vec{x}_\beta \tag{1.62}$$

Generalization

β ... NOT a dual basis

$\vec{b}_i(\vec{b}_j) \neq I$

The value of $\vec{h} \in L^\star$ on $\vec{x} \in L$ is obtained by multiplying $\vec{x}_\beta$ by the transpose of $\vec{h}_{\beta^\star}$ from the left.

Notice that the middle matrix on the right in Equation 1.59 evaluates into the identity. This is the consequence of using the pair of a basis and its dual basis. The formula 1.62 can be generalized to the situation when bases are not dual by evaluating the middle matrix accordingly. In general

$$\vec{h}(\vec{x}) = \vec{h}_{\bar{\beta}}^\top\,[\vec{b}_i(\vec{b}_j)]\,\vec{x}_\beta \tag{1.63}$$

where matrix $[\vec{\bar{b}}_i(\vec{b}_j)]$ is constructed from the respective bases $\beta, \bar{\beta}$ of L and $L^\star$.

§2 Changing the basis in a linear space and in its dual Let us now look at what happens with coordinates of vectors of $L^\star$ when passing from the dual basis $\beta^\star$ to the dual basis $\beta'^\star$ induced by passing from a basis β to a basis β' in L. Consider vector $\vec{x} \in L$ and a linear function $\vec{h} \in L^\star$ and their coordinates $\vec{x}_\beta, \vec{x}_{\beta'}$ and $\vec{h}_{\beta^\star}, \vec{h}_{\beta'^\star}$ w.r.t. the respective bases. Introduce further matrix $\mathtt{A}$ transforming coordinates of vectors in L as

$$\vec{x}_{\beta'} = \mathtt{A}\,\vec{x}_\beta \tag{1.64}$$

when passing from β to β'.

Basis $\beta^\star$ is the dual basis to β and basis $\beta'^\star$ is the dual basis to β' and therefore

$$\beta \quad \vec{h}_{\beta^\star}^\top\,\vec{x}_\beta = \vec{h}(\vec{x}) = \vec{h}_{\beta'^\star}^\top\,\vec{x}_{\beta'} \quad \beta' \tag{1.65}$$

for all $\vec{x} \in L$ and all $\vec{h} \in L^\star$. Hence

$$\vec{h}_{\beta^\star}^\top\,\cancel{\vec{x}_\beta} = \vec{h}_{\beta'^\star}^\top\,\mathtt{A}\,\cancel{\vec{x}_\beta} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.66}$$

for all $\vec{x} \in L$ and therefore

$$\boxed{\vec{h}_{\beta^\star}^\top = \vec{h}_{\beta'^\star}^\top\,\mathtt{A}} \tag{1.67}$$

or equivalently

$$\boxed{\vec{h}_{\beta^\star} = \mathtt{A}^\top\vec{h}_{\beta'^\star}} \tag{1.68}$$

Let us now see what is the meaning of the rows of matrix $\mathtt{A}$. It becomes clear from Equation 1.67 that the columns of matrix $\mathtt{A}^\top$ can be viewed as vectors of coordinates of basic vectors of $\beta'^\star = [\vec{b}_1'^\star, \vec{b}_2'^\star, \vec{b}_3'^\star]$ in the basis

Coupled (β, β^*) (dual to each other)

$(L^*)^* \leftarrow$ fun fun LS $\cong L$

fun LS

Lin. S.

$$\begin{array}{lcc} L\,(\mathbb{R}^n) & \vec{x}_\beta & \xrightarrow{\ A\ } & \vec{x}_{\beta'} \\ & \downarrow & & \downarrow \\ L^*\,(\mathbb{R}^n) & \vec{h}_{\beta^*} & \xrightarrow{\ A^{-T}\ } & \vec{h}_{\beta'^*} \end{array}$$

dual to β dual to β'

$$A \longrightarrow A^T$$

$$\beta \xrightarrow{\quad} \beta' \qquad \beta'^* \xrightarrow{\quad} \beta^*$$

$\beta^\star = [\vec{b}_1^\star, \vec{b}_2^\star, \vec{b}_3^\star]$ and therefore

$$\boxed{\vec{h}_{\beta^\star} = \mathtt{A}^\top \vec{h}_{\beta'^\star}} \quad \mathtt{A} = \begin{bmatrix} \vec{b}'^{\star\top}_{1\beta^\star} \\ \vec{b}'^{\star\top}_{2\beta^\star} \\ \vec{b}'^{\star\top}_{3\beta^\star} \end{bmatrix} \tag{1.69}$$

$$\mathtt{A} = \begin{bmatrix} \vec{b}_{1\beta'}, & \vec{b}_{2\beta'}, & \vec{b}_{3\beta'} \end{bmatrix}$$

? what is in the columns?

$$\vec{x}_{\beta'} = \mathtt{A}\,\vec{x}_\beta$$

$$\vec{h}_{\beta'^\star} = \mathtt{A}^{-\top}\,\vec{h}_{\beta^\star}$$

which means that the rows of A are coordinates of the dual basis of the primed dual space in the dual basis of the non-primed dual space.

Finally notice that we can also write

$$\vec{h}_{\beta'^\star} = \mathtt{A}^{-\top}\vec{h}_{\beta^\star} \tag{1.70}$$

which is formally identical with Equation 1.15.

$\approx$

§ 3 When do coordinates transform the same way in a basis and in its dual basis It is natural to ask when it happens that the coordinates of linear functions in $L^\star$ w.r.t. the dual basis $\beta^\star$ transform the same way as the coordinates of vectors of L w.r.t. the original basis β, i.e.

$$\left.\begin{aligned} \vec{x}_{\beta'} &= \mathtt{A}\,\vec{x}_\beta \quad (1.71) \\ \vec{h}_{\beta'^\star} &= \mathtt{A}\,\vec{h}_{\beta^\star} \quad (1.72) \end{aligned}\right\} \text{special situation}$$

What is in the rows o A ?

for all $\vec{x} \in L$ and all $\vec{h} \in L^\star$. Considering Equation 1.70, we get

$$\mathtt{A} = \mathtt{A}^{-\top} \tag{1.73}$$

$$\boxed{\mathtt{A}^\top\mathtt{A} = \mathtt{I}} \quad \in \mathbb{R}^{3\times 3} \quad \text{orthogonal} \tag{1.74}$$

(|A| = 1 orthonormal

cols $\equiv$ β, L

rows $\equiv$ $\beta^\star$, $L^\star$

Notice that this is, for instance, satisfied when A is a rotation [2]. In such a case, one often does not anymore distinguish between vectors of L and $L^\star$ because they behave the same way and it is hence possible to represent linear functions from $L^\star$ by vectors of L.

§ 4 Coordinates of the basis dual to a general basis We denote the standard basis in $\mathbb{R}^3$ by σ and its dual (standard) basis in $\mathbb{R}^{3\star}$ by $\sigma^\star$. Now, we can further establish another basis $\gamma = \begin{bmatrix}\vec{c}_1 & \vec{c}_2 & \vec{c}_3\end{bmatrix}$ in $\mathbb{R}^3$ and its dual basis $\gamma^\star = \begin{bmatrix}\vec{c}_1^{\,\star} & \vec{c}_2^{\,\star} & \vec{c}_3^{\,\star}\end{bmatrix}$ in $\mathbb{R}^{3\star}$. We would like to find the coordinates $\gamma^\star_{\sigma^\star} = \begin{bmatrix}\vec{c}^{\,\star}_{1\sigma^\star} & \vec{c}^{\,\star}_{2\sigma^\star} & \vec{c}^{\,\star}_{3\sigma^\star}\end{bmatrix}$ of vectors of $\gamma^\star$ w.r.t. $\sigma^\star$ as a function of coordinates $\gamma_\sigma = \begin{bmatrix}\vec{c}_{1\sigma} & \vec{c}_{2\sigma} & \vec{c}_{3\sigma}\end{bmatrix}$ of vectors of γ w.r.t. σ.

Considering Equations 1.55 and 1.62 we are getting

$$\vec{c}^{\,\star\top}_{i\sigma^\star}\vec{c}_{j\sigma} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \quad \text{for } i, j = 1, 2, 3 \tag{1.75}$$

γ, σ

which can be rewritten in a matrix form as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \vec{c}^{\,\star\top}_{1\sigma^\star} \\ \vec{c}^{\,\star\top}_{2\sigma^\star} \\ \vec{c}^{\,\star\top}_{3\sigma^\star} \end{bmatrix} \begin{bmatrix}\vec{c}_{1\sigma} & \vec{c}_{2\sigma} & \vec{c}_{3\sigma}\end{bmatrix} = \gamma^{\star\top}_{\sigma^\star}\,\gamma_\sigma \tag{1.76}$$

and therefore

$$\gamma^\star_{\sigma^\star} = \gamma_\sigma^{-\top} \tag{1.77}$$

finite dim L

§ 5 Remark on higher dimensions We have introduced the dual space and the dual basis in a three-dimensional linear space. The definition of the dual space is exactly the same for any linear space. The definition of the dual basis is the same for all finite-dimensional linear spaces [1]. For any n-dimensional linear space L and its basis β, we get the corresponding n-dimensional dual space $L^\star$ with the dual basis $\beta^\star$.

dim L = dim L*

L ≡ polynomials
real fns
may not be true
i.e. dim L ≠ dim L*

1.5 Operations with matrices

Matrices are a powerful tool which can be used in many ways. Here we review a few useful rules for matrix manipulation. The rules are often studied in multi-linear algebra and tensor calculus. We shall not

Notice, that *in the real projective plane there is exactly one point incident to two distinct lines*.

This is not true in an affine plane because there are (parallel) lines that have no point in common!

2.3 Line coordinates under homography

Let us now investigate the behavior of homogeneous coordinates of lines in projective plane mapped by a homography.

Let us have two points represented by vectors $\vec{x}_\beta$, $\vec{y}_\beta$. We now map the points, represented by vectors $\vec{x}_\beta$, $\vec{y}_\beta$, by a homography, represented by matrix $\mathtt{H}$, to points represented by vectors $\vec{x}'_{\beta'}$, $\vec{y}'_{\beta'}$ such that there are $\lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1\lambda_2 \neq 0$

points $\vec{x}$, $\vec{y}$

$$\lambda_1\, \vec{x}'_{\beta'} = \mathtt{H}\, \vec{x}_\beta \tag{2.27}$$

$$\lambda_2\, \vec{y}'_{\beta'} = \mathtt{H}\, \vec{y}_\beta \tag{2.28}$$

Homogeneous coordinates $\vec{p}_{\bar{\beta}}$ of the line passing through points represented by $\vec{x}_\beta$, $\vec{y}_\beta$ and homogeneous coordinates $\vec{p}'_{\bar{\beta}'}$ of the line passing through points represented by $\vec{x}'_{\beta'}$, $\vec{y}'_{\beta'}$ are obtained by solving the linear systems

$$\vec{p}_{\bar{\beta}}^{\top}\, \vec{x}_\beta = 0 \qquad \text{and} \qquad {\vec{p}'_{\bar{\beta}'}}^{\top}\, \vec{x}'_{\beta'} = 0 \tag{2.29}$$

$$\vec{p}_{\bar{\beta}}^{\top}\, \vec{y}_\beta = 0 \qquad\qquad {\vec{p}'_{\bar{\beta}'}}^{\top}\, \vec{y}'_{\beta'} = 0 \tag{2.30}$$

for a non-trivial solutions. Writing down the incidence of points and lines, we get

$$\lambda_1\, \vec{p}_{\bar{\beta}}^{\top}\, \mathtt{H}^{-1}\, \vec{x}'_{\beta'} = 0 \quad \Leftrightarrow \quad \vec{p}_{\bar{\beta}}^{\top}\, \mathtt{H}^{-1}\, \vec{x}'_{\beta'} = 0$$

$$\lambda_2\, \vec{p}_{\bar{\beta}}^{\top}\, \mathtt{H}^{-1}\, \vec{y}'_{\beta'} = 0 \quad \Leftrightarrow \quad \vec{p}_{\bar{\beta}}^{\top}\, \mathtt{H}^{-1}\, \vec{y}'_{\beta'} = 0$$

We see that $\vec{p}'_{\bar{\beta}'}$, and $\mathtt{H}^{-\top}\vec{p}_{\bar{\beta}}$ are solutions of the same set of homogeneous equations. When $\vec{x}_\beta$, $\vec{y}_\beta$ are independent, then there is $\lambda \in \mathbb{R}$ such that

$$\boxed{\lambda\, \vec{p}'_{\bar{\beta}'} = \mathtt{H}^{-\top}\vec{p}_{\bar{\beta}}} \tag{2.31}$$

since the solution space is one-dimensional. Equation 2.31 gives the relationship between homogeneous coordinates of a line and its image under homography $\mathtt{H}$.

$$\vec{x}_{\beta'} = A\, \vec{x}_{\beta}$$

$$\vec{h}_{\beta'^*} = A^{-T}\, \vec{h}_{\beta^*}$$

2.3.1 Join under homography

Let us go one step further and establish formulas connecting line coordinates constructed by vector products. Construct joins as

$$\vec{p}_{\bar{\beta}} = \vec{x}_\beta \times \vec{y}_\beta \quad \text{and} \quad \vec{p}'_{\bar{\beta}'} = \vec{x}'_{\beta'} \times \vec{y}'_{\beta'} \tag{2.32}$$

$$P_{\bar{\beta}} \equiv P_{\beta^*}$$

$$\bar{\beta} \qquad \beta^*$$

and use Equation 1.45 to get

$$\vec{x}_{\beta'} \times \vec{y}_{\beta'} = \frac{\mathtt{H}^{-\top}}{|\mathtt{H}^{-\top}|}(\vec{x}_\beta \times \vec{y}_\beta) \tag{2.33}$$

$$(\lambda_1\, \vec{x}'_{\beta'}) \times (\lambda_2\, \vec{y}'_{\beta'}) = \frac{\mathtt{H}^{-\top}}{|\mathtt{H}^{-\top}|}(\vec{x}_\beta \times \vec{y}_\beta) \tag{2.34}$$

$$\vec{x}'_{\beta'} \times \vec{y}'_{\beta'} = \frac{\mathtt{H}^{-\top}}{\lambda_1\, \lambda_2\, |\mathtt{H}^{-\top}|}(\vec{x}_\beta \times \vec{y}_\beta) \tag{2.35}$$

$$\vec{p}'_{\bar{\beta}'} = \frac{\mathtt{H}^{-\top}}{\lambda_1\, \lambda_2\, |\mathtt{H}^{-\top}|}\vec{p}_{\bar{\beta}} \tag{2.36}$$

2.3.2 Meet under homography

Let us next look at the meet. Let point $\vec{x}$ be the meet of lines $\vec{p}$ and $\vec{q}$ with line cordinates $\vec{p}_{\bar{\beta}}$, $\vec{q}_{\bar{\beta}}$, which are related by a homography $\mathtt{H}$ to line coordinates $\vec{p}\,'_{\bar{\beta}'}$ and $\vec{q}\,'_{\bar{\beta}'}$ by

$$\lambda_1\, \vec{p}\,'_{\bar{\beta}'} = \mathtt{H}^{-\top}\, \vec{p}_{\bar{\beta}} \tag{2.37}$$

$$\lambda_2\, \vec{q}\,'_{\bar{\beta}'} = \mathtt{H}^{-\top}\, \vec{q}_{\bar{\beta}} \tag{2.38}$$

for some non-zero λ_1, λ_2. Construct meets as

$$\vec{x}_\beta = \vec{p}_{\bar{\beta}} \times \vec{q}_{\bar{\beta}} \quad \text{and} \quad \vec{x}\,'_{\beta'} = \vec{p}\,'_{\bar{\beta}'} \times \vec{q}\,'_{\bar{\beta}'} \tag{2.39}$$

and, similarly as above, use Equation 1.45 to get

$$\vec{x}\,'_{\beta'} = \frac{(\mathtt{H}^{-\top})^{-\top}}{\lambda_1\,\lambda_2\;|(\mathtt{H}^{-\top})^{-\top}|}\,\vec{x}_\beta = \frac{\mathtt{H}}{\lambda_1\,\lambda_2\;|\mathtt{H}|}\,\vec{x}_\beta \tag{2.40}$$

2.3.3 Meet of join under homography

We can put the above together to get meet of join under homography. We consider two pairs of points represented by their homogeneous coordinates $\vec{x}_\beta$, $\vec{y}_\beta$, and $\vec{z}_\beta$, $\vec{w}_\beta$ and the corresponding pairs of points with their homogeneous coordinates $\vec{x}\,'_{\beta'}$, $\vec{y}\,'_{\beta'}$, and $\vec{z}\,'_{\beta'}$, $\vec{w}\,'_{\beta'}$ related by homography $\mathtt{H}$ as

$$\lambda_1\,\vec{x}\,'_{\beta'} = \mathtt{H}\,\vec{x}_\beta\,, \quad \lambda_2\,\vec{y}\,'_{\beta'} = \mathtt{H}\,\vec{y}_\beta\,, \quad \lambda_3\,\vec{z}\,'_{\beta'} = \mathtt{H}\,\vec{z}_\beta\,, \quad \lambda_4\,\vec{w}\,'_{\beta'} = \mathtt{H}\,\vec{w}_\beta \tag{2.41}$$

Let us now consider point

$$\vec{v}\,'_{\beta'} = (\vec{x}\,'_{\beta'} \times \vec{y}\,'_{\beta'}) \times (\vec{z}\,'_{\beta'} \times \vec{w}\,'_{\beta'}) \tag{2.42}$$

$$= \left(\frac{\mathtt{H}^{-\top}}{\lambda_1\,\lambda_2\;|\mathtt{H}^{-\top}|}(\vec{x}_\beta \times \vec{y}_\beta)\right) \times \left(\frac{\mathtt{H}^{-\top}}{\lambda_3\,\lambda_4\;|\mathtt{H}^{-\top}|}(\vec{z}_\beta \times \vec{w}_\beta)\right) \tag{2.43}$$

$$= \frac{\mathtt{H}\,|\mathtt{H}|}{\lambda_1\,\lambda_2\,\lambda_3\,\lambda_4}(\vec{x}_\beta \times \vec{y}_\beta) \times (\vec{z}_\beta \times \vec{w}_\beta) \tag{2.44}$$

$$= \frac{\mathtt{H}\,|\mathtt{H}|}{\lambda_1\,\lambda_2\,\lambda_3\,\lambda_4}\vec{v}_\beta \tag{2.45}$$