

has a special form which corresponds to a special change of a coordinate system in the three-dimensional affine space.

3.4 Reconstruction from two calibrated views

Let us further assume that camera calibration matrices K_1, K_2 are known. Hence we can pass from F to E using Equations 3.14, 3.15 as

$$E = K_2^T F K_1 \tag{3.41}$$

then recover the relative pose of the cameras, set their coordinate systems and finally reconstruct points of the scene.

3.4.1 Camera computation

To simplify the setting, we will first pass from "uncalibrated" image points $\vec{x}_{1\beta_1}, \vec{x}_{2\beta_2}$ using K_1, K_2 to "calibrated"

$$\vec{x}_{1\gamma_1} = K_1^{-1} \vec{x}_{1\beta_1} \quad \text{and} \quad \vec{x}_{2\gamma_2} = K_2^{-1} \vec{x}_{2\beta_2} \tag{3.42}$$

and then use camera projection matrices as follows

$$\zeta_1 \vec{x}_{1\gamma_1} = P_{1\gamma_1} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \quad \text{and} \quad \zeta_2 \vec{x}_{2\gamma_2} = P_{2\gamma_2} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \tag{3.43}$$

Matrix H allows us to choose the global coordinate system of the scene as (C_1, e_1) . Setting

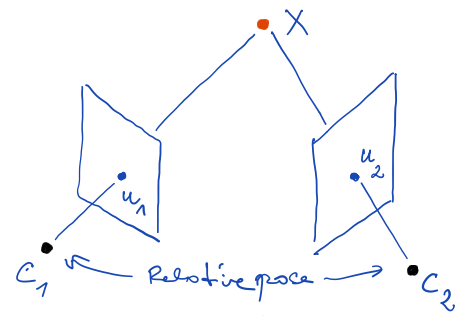
$$H^{-1} = \begin{bmatrix} R_{1\delta}^T & \vec{C}_{1\delta} \\ 0^T & 1 \end{bmatrix} \quad \checkmark$$

$$P_{1\gamma_1} = [I \mid 0]$$

Calibrated
 K_1, K_2 known
 \Downarrow
 $\vec{x}_{i\beta_i} \rightarrow \vec{x}_{i\gamma_i}$

(3.44)

3D reconstruction by triangulation



$$u_1, u_2 \xrightarrow{?} X$$

$$\begin{bmatrix} \vec{u}_{1d_1} \\ \vec{u}_{2d_2} \end{bmatrix} \longrightarrow \vec{X}_\delta$$

$$\mathbb{R}^4 \longrightarrow \mathbb{R}^3$$

Essential matrix

$$E = R_3 [C_{2\delta} - C_{1\delta}]_x R_1^T$$

Fundamental matrix

$$F = K_2^{-T} E K_1^{-1}$$

we get from Equation 3.38

$$P_{1\gamma_1} = [I | \vec{0}] \quad (3.45)$$

$$P_{2\gamma_2} = [R_2 R_1^T | -R_2 (\vec{C}_{2\delta} - \vec{C}_{1\delta})] = [R_2 R_1^T | -R_2 R_1^T (\vec{C}_{2e_1} - \vec{C}_{1e_1})] \quad (3.46)$$

$$= [R | -R \vec{C}_{e_1}] \quad (3.47)$$

and the corresponding essential matrix

$$E = R \begin{bmatrix} \vec{C}_{e_1} \\ \times \end{bmatrix}_x \quad (3.48)$$

From image measurements, $\vec{x}_{1\gamma_1}, \vec{x}_{2\gamma_2}$, we can compute, Section 3.2 matrix

Computed up to scale: $G = \tau E = \tau R \begin{bmatrix} \vec{C}_{e_1} \\ \times \end{bmatrix}_x \quad (3.49)$

and hence we can get E only up to a non-zero multiple τ . Therefore, we can recover \vec{C}_{e_1} only up to τ .

We will next fix τ up to its sign s_1 . Consider that the Frobenius norm of a matrix G

$$\| \cdot \|_F = \| \cdot \|_2 = \sqrt{\lambda_1 + \lambda_2 + \lambda_3}$$

$$\|G\|_F = \sqrt{\sum_{i,j=1}^3 G_{ij}^2} = \sqrt{\text{trace}(G^T G)} = \sqrt{\text{trace}(\tau^2 \begin{bmatrix} \vec{C}_{e_1} \\ \times \end{bmatrix}_x^T R^T R \begin{bmatrix} \vec{C}_{e_1} \\ \times \end{bmatrix}_x)} \quad (3.50)$$

$$= \sqrt{\tau^2 \text{trace}(\begin{bmatrix} \vec{C}_{e_1} \\ \times \end{bmatrix}_x^T \begin{bmatrix} \vec{C}_{e_1} \\ \times \end{bmatrix}_x)} \quad (3.51)$$

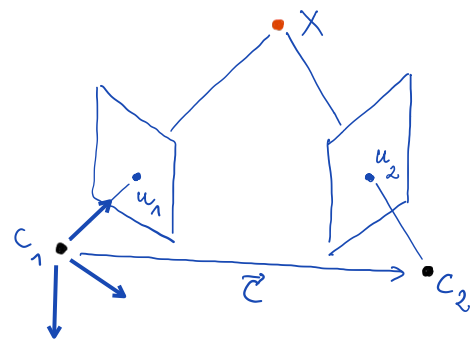
$$= |\tau| \sqrt{2 \|\vec{C}_{e_1}\|^2} = |\tau| \sqrt{2} \|\vec{C}_{e_1}\| \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$y^2 + z^2 + x^2 + z^2 + x^2 + y^2 = 2 \|\vec{C}_{e_1}\|^2$$

We have used the following identities

$$G^T G = \tau^2 \begin{bmatrix} \vec{C}_{e_1} \\ \times \end{bmatrix}_x^T R^T R \begin{bmatrix} \vec{C}_{e_1} \\ \times \end{bmatrix}_x = \tau^2 \begin{bmatrix} \vec{C}_{e_1} \\ \times \end{bmatrix}_x^T \begin{bmatrix} \vec{C}_{e_1} \\ \times \end{bmatrix}_x \quad (3.52)$$

$$= \tau^2 \begin{bmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{bmatrix} \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} = \tau^2 \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & yz \\ -xz & -yz & x^2 + y^2 \end{bmatrix}$$



$$(0, \delta) := (c_1, \epsilon_1)$$

The world coordinate system in the first camera

$$P_{1\gamma_1} = [R_1 | -R_1 \vec{C}_{1\delta}] \begin{bmatrix} R_1^T & \vec{C}_{1\delta} \\ 0 & 1 \end{bmatrix} = [I | 0] \quad H^{-1} = \begin{bmatrix} R_1^T & \vec{C}_{1\delta} \\ 0^T & 1 \end{bmatrix}$$

$$P_{2\gamma_2} = [R_2 | -R_2 \vec{C}_{2\delta}] \begin{bmatrix} R_1^T & \vec{C}_{1\delta} \\ 0 & 1 \end{bmatrix} = [R_2 R_1^T | -R_2 R_1^T \vec{C}_{1e_1}] = [R | -R \vec{C}_{e_1}]$$

We can now construct normalized matrix \bar{G} as

$$\bar{G} = \frac{\sqrt{2}G}{\sqrt{\sum_{i,j=1}^3 G_{ij}^2}} = \frac{\tau}{|\tau|} R \begin{bmatrix} \vec{c}_{e_1} \\ \|\vec{c}_{e_1}\| \end{bmatrix}_x = s_1 R [\vec{t}_{e_1}]_x = \pm E \quad (3.53)$$

with new unknown $s_1 \in \{+1, -1\}$ and \vec{t}_{e_1} denoting the unit vector in the direction of the second camera center in e_1 basis.

We can find vector $\vec{v}_{e_1} = s_2 \vec{t}_{e_1}$ with new unknown $s_2 \in \{+1, -1\}$ by solving

$$\bar{G} \vec{v}_{e_1} = 0 \quad \text{subject to} \quad \|\vec{v}_{e_1}\| = 1 \quad (3.54)$$

to get

$$\bar{G} = s_1 R \begin{bmatrix} 1 \\ s_2 \vec{v}_{e_1} \end{bmatrix}_x = \frac{s_1}{s_2} R [\vec{v}_{e_1}]_x \quad (3.55)$$

$$s \bar{G} = R [\vec{v}_{e_1}]_x \quad \leftarrow 1/s \quad (3.56)$$

$$3 \times 3 \quad [s g_1 \quad s g_2 \quad s g_3] = R [v_1 \quad v_2 \quad v_3] \quad 3 \times 3 \quad (3.57)$$

with unknown $s \in \{+1, -1\}$, unknown rotation R and known matrices $[g_1 \quad g_2 \quad g_3] = \bar{G}$ and $[v_1 \quad v_2 \quad v_3] = [\vec{v}_{e_1}]_x$.

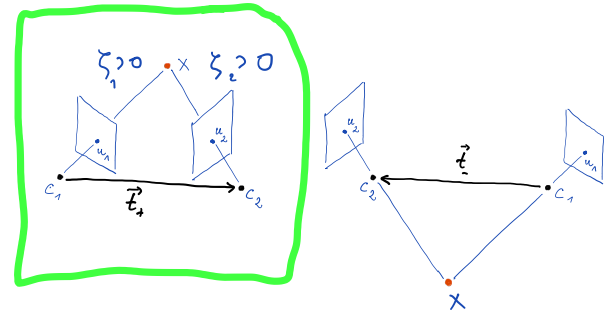
This is a matricial equation. Matrices $\bar{G}, [\vec{v}_{e_1}]_x$ are of rank two and hence do not determine R uniquely unless we use $R^T R = I$ and $|R| = 1$. That leads to a set of polynomial equations. They can be solved but we will use the property of vector product, §2 to directly construct regular matrices that will determine R uniquely for a fixed s .

Consider that for every regular $A \in \mathbb{R}^{3 \times 3}$, we have, §2

$$(A \vec{x}_\beta) \times (A \vec{y}_\beta) = \vec{x}_{\beta'} \times \vec{y}_{\beta'} = \frac{A^{-T}}{|A^{-T}|} (\vec{x}_\beta \times \vec{y}_\beta) \quad (3.58)$$

which for R gives

$$(R \vec{x}_\beta) \times (R \vec{y}_\beta) = R (\vec{x}_\beta \times \vec{y}_\beta) \quad (3.59)$$



rank $\bar{G} = 2$
rank $[\vec{v}_{e_1}]_x = 2$

3x3 = 9 eqns.
linear
Rank x 9 + 7 eq.

$$s \bar{G} = R [\vec{v}_{e_1}]_x$$

↓ ?
R

deg², deg³
→ 6 + 1 = 7

$$R \text{ rotation} \hat{=} R^T R = I, \det R = 1$$

non-linear

Using it for $i, j = 1, 2, 3$ to get

$$(s \mathbf{g}_i) \times (s \mathbf{g}_j) = (\mathbf{R} \mathbf{v}_i) \times (\mathbf{R} \mathbf{v}_j) \quad (3.60)$$

$$1 = (\frac{1}{s})^2 \quad \cancel{s^2} (\mathbf{g}_i \times \mathbf{g}_j) = \mathbf{R}(\mathbf{v}_i \times \mathbf{v}_j) \quad (3.61)$$

$$\underline{(\mathbf{g}_i \times \mathbf{g}_j) = \mathbf{R}(\mathbf{v}_i \times \mathbf{v}_j)} \quad (3.62)$$

$$\left\{ \begin{aligned} \vec{g}_1 \times \vec{g}_2 &= \mathbf{R}(\vec{v}_1 \times \vec{v}_2) \\ \vec{g}_2 \times \vec{g}_3 &= \mathbf{R}(\vec{v}_2 \times \vec{v}_3) \\ \vec{g}_1 \times \vec{g}_3 &= \mathbf{R}(\vec{v}_1 \times \vec{v}_3) \end{aligned} \right.$$

i.e. three more vector equations. Notice how s disappeared in the vector product.

We see that we can write

$$\begin{bmatrix} s \mathbf{g}_1 & s \mathbf{g}_2 & s \mathbf{g}_3 & \mathbf{g}_1 \times \mathbf{g}_2 & \mathbf{g}_2 \times \mathbf{g}_3 & \mathbf{g}_1 \times \mathbf{g}_3 \end{bmatrix} = \mathbf{R}_s \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_1 \times \mathbf{v}_2 & \mathbf{v}_2 \times \mathbf{v}_3 & \mathbf{v}_1 \times \mathbf{v}_3 \end{bmatrix} \quad (3.63)$$

There are two solutions \mathbf{R}_+ for $s = +1$ and \mathbf{R}_- for $s = -1$. We can next compute two solutions $\vec{t}_{+\epsilon_1} = +\vec{v}_{\epsilon_1}$ and $\vec{t}_{-\epsilon_1} = -\vec{v}_{\epsilon_1}$ and combine them together to four possible solutions

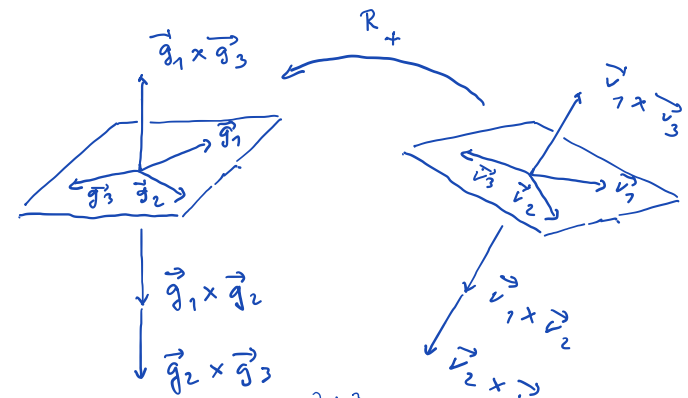
$$\mathbf{P}_{2\gamma_2++} = \underline{\mathbf{R}_+} \left[\mathbf{I} \mid -\underline{\vec{t}_{+\epsilon_1}} \right] \quad (3.64)$$

$$\mathbf{P}_{2\gamma_2+-} = \underline{\mathbf{R}_+} \left[\mathbf{I} \mid -\underline{\vec{t}_{-\epsilon_1}} \right] \quad (3.65)$$

$$\mathbf{P}_{2\gamma_2-+} = \underline{\mathbf{R}_-} \left[\mathbf{I} \mid -\underline{\vec{t}_{+\epsilon_1}} \right] \quad (3.66)$$

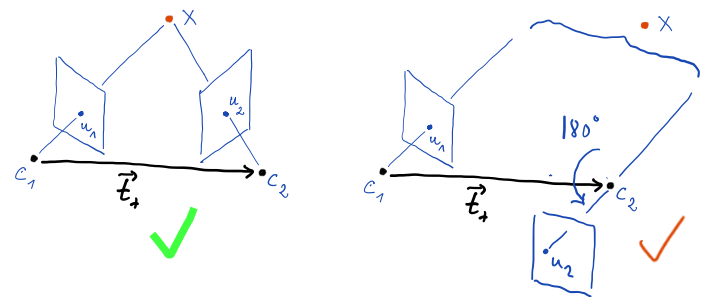
$$\mathbf{P}_{2\gamma_2--} = \underline{\mathbf{R}_-} \left[\mathbf{I} \mid -\underline{\vec{t}_{-\epsilon_1}} \right] \quad (3.67)$$

The above four camera projection matrices are compatible with $\bar{\mathbf{G}}$. The one which corresponds to the actual matrix can be selected by requiring that all reconstructed points lie in front of the cameras, i.e. that the reconstructed points are all positive multiples of vectors $\vec{x}_{1\epsilon_1}$ and $\vec{x}_{2\epsilon_2}$ for all image points.



$$E = \mathbf{R}[\vec{t}]_x = \mathbf{R} \left(-\mathbf{I} + 2 \frac{\vec{t} \vec{t}^T}{\vec{t}^T \vec{t}} \right) [\vec{t}]_x$$

rotation by 180°



3.4.2 Point computation

Let us assume having camera projection matrices P_1, P_2 and image points $\vec{x}_{1\beta_1}, \vec{x}_{2\beta_2}$ such that

$$\zeta_1 \vec{x}_{1\beta_1} = \underbrace{P_1}_{K_1} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \quad \text{and} \quad \zeta_2 \vec{x}_{2\beta_2} = \underbrace{P_2}_{K_2} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \quad (3.68)$$

4 cases

We can get \vec{X}_δ , and ζ_1, ζ_2 by solving the following system of (inhomogeneous) linear equations

$$\begin{matrix} 1 & 1 & 4 \\ 3 & \begin{bmatrix} \vec{x}_{1\beta_1} & \vec{0} & -P_1 \\ \vec{0} & \vec{x}_{2\beta_2} & -P_2 \end{bmatrix} & \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vec{X}_\delta \\ \mathbf{1} \end{bmatrix} = 0 \end{matrix} \quad (3.69)$$

6 x 6 *read out*

Projection equations

Solve this linear system
for $\zeta_1, \zeta_2, \vec{X}_\delta \in \mathbb{R}^3$

3.5 Calibrated relative camera pose computation

In the previous chapter, we had first computed a multiple of the fundamental matrix from seven point correspondences and only then used camera calibration matrices to recover a multiple of the essential matrix. Here we will use the camera calibration right from the beginning to obtain a multiple of the essential matrix directly from only five image correspondences. Not only that five is smaller than seven but using the calibration right from the beginning permits all points of the scene generating the correspondences to lie in a plane.

We start from Equation 3.42 to get $\vec{x}_{1\gamma_1}$ and $\vec{x}_{2\gamma_2}$ from Equation 3.43 which are related by

$$\vec{x}_{2\beta_2}^\top K_2^{-\top} E K_1^{-1} \vec{x}_{1\beta_1} = 0 \quad (3.70)$$

$$\vec{x}_{2\gamma_2}^\top E \vec{x}_{1\gamma_1} = 0 \quad (3.71)$$

The above equation holds true for all pairs of image points $(\vec{x}_{1\gamma_1}, \vec{x}_{2\gamma_2})$ that are in correspondence, i.e. are projections of the same point of the scene.

$P_1 = [I | 0] \quad P_{2++} \rightarrow X_{++}$
4 cases
 $- | - \quad P_{2+-} \rightarrow X_{+-}$
 $- | - \quad P_{2-+} \rightarrow X_{-+}$
 $- | - \quad P_{2--} \rightarrow X_{--}$

$\zeta_1 \vec{x}_{1\gamma_1} = P_1 X \quad \times$
 $\zeta_2 \vec{x}_{2\gamma_2} = P_{2ab} X \quad \times$