# Sequential decisions under uncertainty
# Policy iteration

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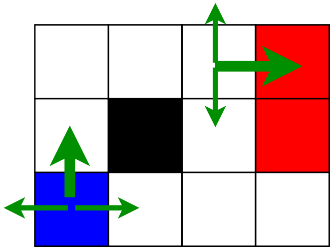
**Notes**

# Unreliable actions in observable grid world

- Walls block movement – agent/robot stays in place.
- Actions do not always go as planned.
- Agent receives rewards each time step:
  - Small "living" reward/penalty.
  - Big rewards/penalties at the end.
- Goal: maximize sum of (discounted) rewards





**Notes**

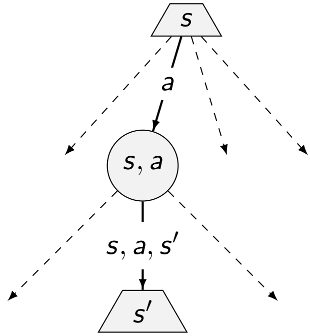
# MDPs recap

## Markov decision processes (MDPs):

- Set of states $\mathcal{S}$
- Set of actions $\mathcal{A}$
- Transitions $p(s'|s,a)$ or $T(s,a,s')$
- Rewards $r(s,a,s')$; and discount $\gamma$

## MDP quantities:

- Policy $\pi(s) : \mathcal{S} \rightarrow \mathcal{A}$
- Utility – sum of (discounted) rewards.
- Values – expected future utility from a state (max-node), $v(s)$
- $Q$-Values – expected future utility from a $q$-state (chance-node), $q(s,a)$



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$Q$-values – like values but given that I have commited to do action $a$ from state $s$.

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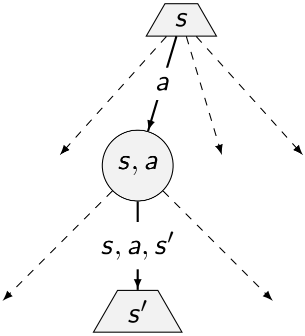




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# Optimal quantities

- The optimal policy: $\pi^*(s)$ – optimal action from state $s$
- Expected utility/return of a policy.

$$U^{\pi}(S_t) = \mathrm{E}^{\pi}\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1}\right]$$

Best policy $\pi^*$ maximizes above.

- The value of a state $s$: $v^*(s)$ – expected utility starting in $s$ and acting optimally.
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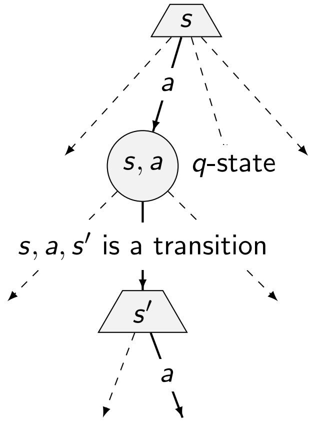




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Remember: Discounted return $G_t$

Returns are successive steps related to each other

$$\begin{aligned} G_t &= R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots \\ &= R_{t+1} + \gamma(R_{t+2} + \gamma^1 R_{t+3} + \gamma^2 R_{t+4} + \cdots) \\ &= R_{t+1} + \gamma G_{t+1} \end{aligned}$$

$G_t \doteq \sum_{k=t+1}^{T} \gamma^{k-t-1} R_k$ including the possibility that $T = \infty$ or $\gamma = 1$, but not both.

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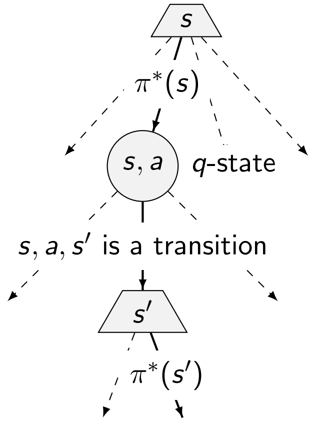


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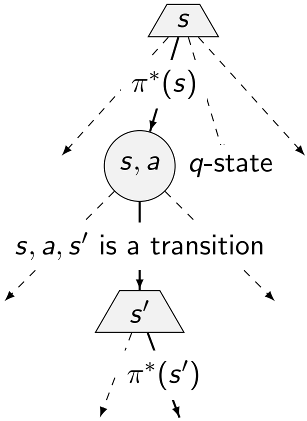




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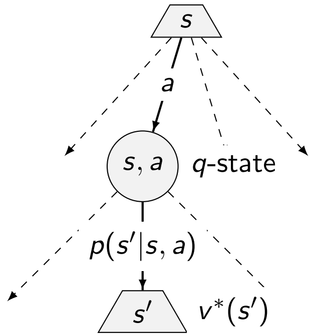
# $V^*$ and $Q^*$

The value of a $q$-state $(s, a)$:

$$q^*(s,a) = \sum_{s'} p(s'|a,s)\left[r(s,a,s') + \gamma\, v^*(s'))\right]$$

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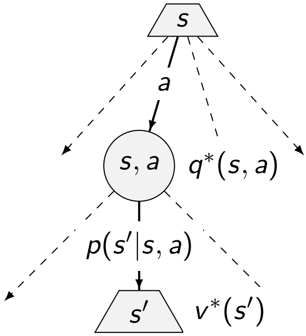




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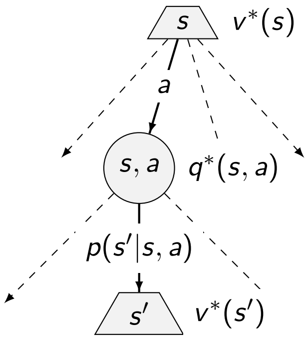





**Notes**

# Maze: $v^*$ vs. $q^*$

| | | | |
|---|---|---|---|
| 0.81 | 0.87 | 0.92 | 1.00 |
| 0.76 | | 0.66 | -1.00 |
| 0.71 | 0.66 | 0.61 | 0.39 |

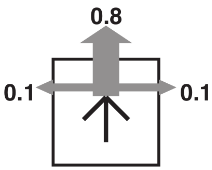
| | | | |
|---|---|---|---|
| 0.78<br>0.77 0.81<br>0.74 | 0.83<br>0.78 0.87<br>0.83 | 0.88<br>0.81 0.92<br>0.68 | 0.00<br>0.00 0.00<br>0.00 |
| 0.76<br>0.72 0.72<br>0.68 | | 0.66<br>0.64 -0.69<br>0.42 | 0.00<br>0.00 0.00<br>0.00 |
| 0.71<br>0.67 0.63<br>0.66 | 0.62<br>0.66 0.58<br>0.62 | 0.59<br>0.61 0.40<br>0.55 | -0.74<br>0.39 0.21<br>0.37 |

$$
\begin{aligned}
q^*(s,a) &= \sum_{s'} p(s'|a,s)\left[r(s,a,s') + \gamma\, v^*(s')\right)\right] \\
v^*(s) &= \max_a q^*(s,a)
\end{aligned}
$$



**Notes**

This is the $R = -0.04$ for nonterminal states maze (AIMA Fig. 17.3).



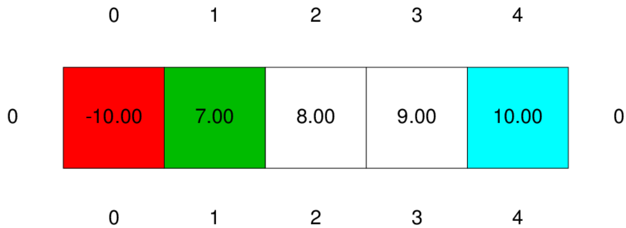

, $\gamma = 1$

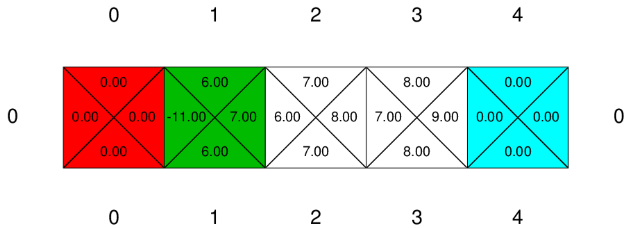
Note that the Value of a state takes into account a number of things:

- the policy – which action will chosen in $s$
- the fact that the goal may be far away and
  - there will be a number of living penalties incured before reaching it
  - the final reward will be discounted
- the transition probabilities

$Q$-values - useful for choosing the best action – getting the policy.

# Maze: $v^*$ vs. $q^*$





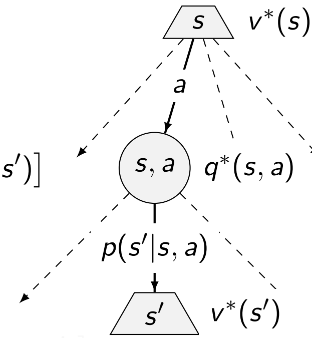

**Notes**

$A = \{\leftarrow, \rightarrow\}$
$P(action - succeeds - as - planned) = 0.8,\ P(reverse - direction - of - movement - than - commanded) = 0.2$

# Value iteration

- Bellman equations **characterize** the optimal values

$$v^*(s) = \max_{a \in A(s)} \sum_{s'} p(s'|s,a)\left[r(s,a,s') + \gamma v^*(s')\right]$$




- Value iteration **computes** them:

$$V_{k+1}(s) \leftarrow \max_{a \in A(s)} \sum_{s'} p(s'|s,a)\left[r(s,a,s') + \gamma V_k(s')\right]$$

Value iteration is a fixed point solution method.



## Notes

Bellman equations:

1. Take correct first action (1 ply of Expectimax)
2. Keep being optimal (recursion $v^*(s')$)

Recall that we may simplify equations by marginalizing rewards if all $r(s,a,s')$ are the same.

$$r(s) = \sum_{s'} p(s'|a,s) r(s,a,s')$$

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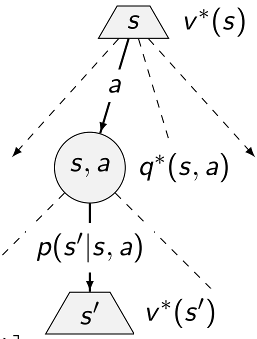


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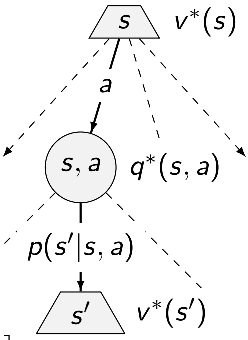


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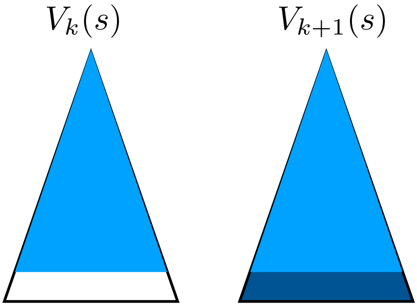
# Convergence

$$V_{k+1}(s) \leftarrow \max_{a \in A(s)} \sum_{s'} p(s'|s,a) \left[ r(s,a,s') + \gamma V_k(s') \right]$$

- Thinking about special cases: deterministic world, $\gamma = 0$, $\gamma = 1$.
- For all $s$, $V_k(s)$ and $V_{k+1}(s)$ can be seen as expectimax search trees of depth $k$ and $k+1$






**Notes**

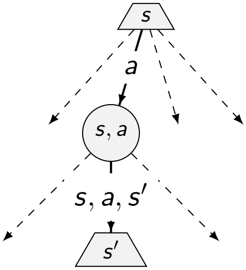
We will show it on the blackboard during the lecture

# From Values to Policy



Notes

# Policy extraction - computing actions from Values




|   | 0 | 1 | 2 | 3 |   |
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- Assume we have $v^*(s)$
- What is the optimal action?
- We need a one-step expectimax:

$$\pi^*(s) = \arg\max_{a \in \mathcal{A}(s)} \sum_{s'} p(s' \mid s, a) \left[ r(s, a, s') + \gamma v^*(s') \right]$$



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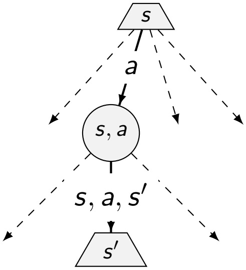


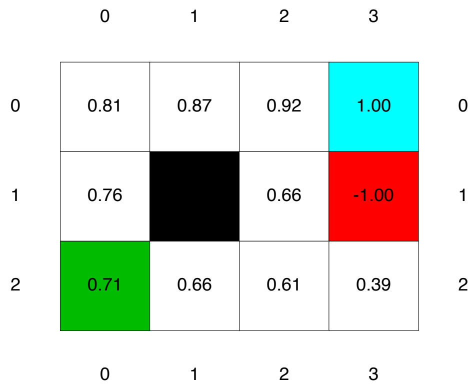


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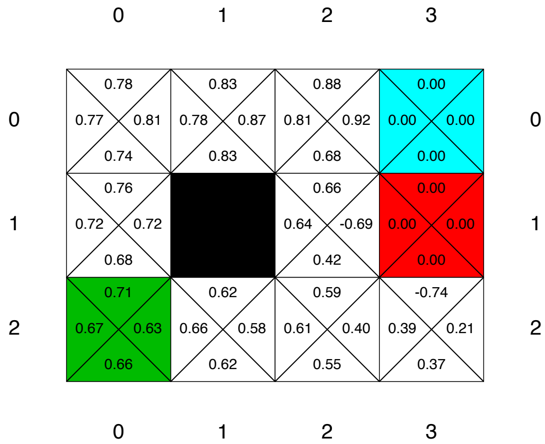
## Notes

# Policy extraction - computing actions from $q$-Values

- Assume we have $q^*(s, a)$
- What is the optimal action?
- Just take the (arg) max:

$$\pi^*(s) = \arg\max_{a \in \mathcal{A}(s)} q^*(s, a)$$

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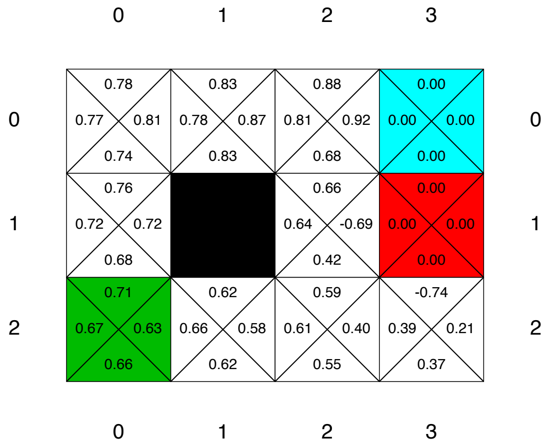


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$$V_{k+1}(s) \leftarrow \max_{a \in \mathcal{A}(s)} \sum_{s'} p(s' \mid s, a) \left[ r(s, a, s') + \gamma V_k(s') \right]$$

- What is complexity of one iteration - over all $S$ states?
- Does the "max" change often?
- When the does the policy converge?
- Can we compute the policy directly?



**Notes**

Complexity: $O(S^2 * A)$ per iteration
For every state (LHS), there can be up to $\sharp S$ also on RHS – if every other state was reachable from the current state.
In addition, all actions from every state need to be considered.
$Max(A)$ **does not** change often.
Policy often converges long before the values.

Run "AIMA Fig. 17.2 / 17.3 demo" with $R = -0.04$
mdp_agents.py, value iteration with $eps = 0.03$, $discount = 0.999999$

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# Policy evaluation

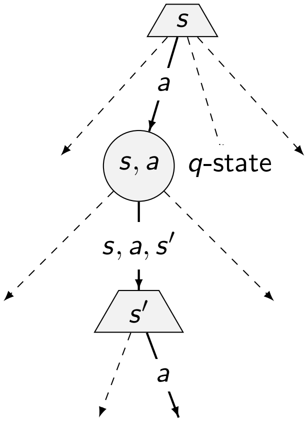
- Assume $\pi(s)$ given.
- How to evaluate (compare)?



**Notes**

Remember last week's quizz?

# Fixed policy, do what $\pi$ says

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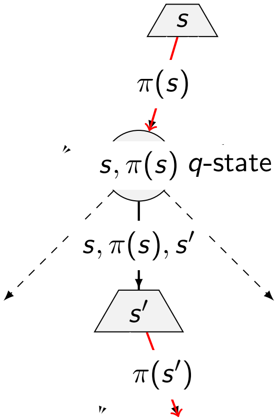


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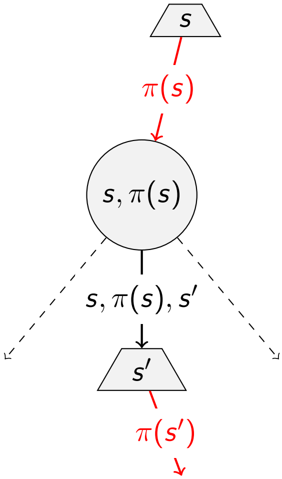
**Notes**

# State values under a fixed policy




- Expectimax trees "max" over all actions ...
- Fixed $\pi$ for each state $\rightarrow$ no "max" operator!

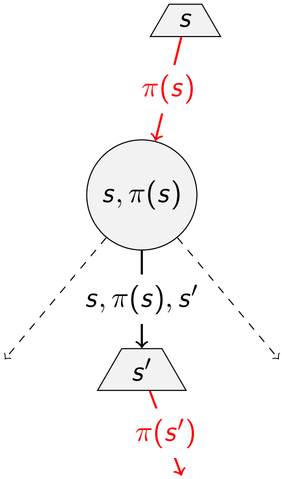
$v^{\pi}(s) = \sum_{s'} p(s' \mid s, \pi(s))[r(s, \pi(s), s') + \gamma v^{\pi}(s')]$



**Notes**

Recall that $v^{\pi}(s)$ quantity contains all the future – expected discounted sum of rewards – executing policy from the state $s$ onwards.

# State values under a fixed policy

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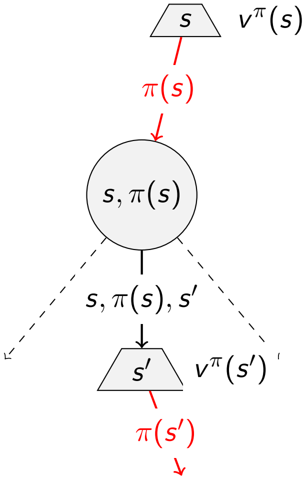


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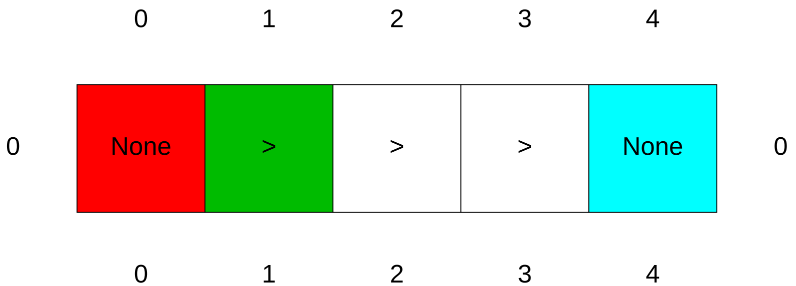
**Notes**

Recall that $v^\pi(s)$ quantity contains all the future – expected discounted sum of rewards – executing policy from the state $s$ onwards.

# How to compute $v^\pi(s)$?

$$v^\pi(s) = \sum_{s'} p(s' \mid s, \pi(s)) \left[ r(s, \pi(s), s') + \gamma v^\pi(s') \right]$$






**Notes**

- by iteration
- solving set of equations

# Policy iteration

- Start with a random policy.
- Step 1: Evaluate it.
- Step 2: Improve it.
- Repeat steps until policy converges.



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**Notes**

# Policy iteration

- Policy $\pi$ evaluation. Solve equations or iterate until convergence.

$$V_{k+1}^{\pi_i}(s) \leftarrow \sum_{s'} p(s' \mid s, \pi(s)) \left[ r(s, \pi(s), s') + \gamma V_k^{\pi_i}(s') \right]$$

- Policy improvement. Look-ahead and keep optimality. Policy extraction from fixed values.

$$\pi_{i+1}(s) = \arg\max_{a \in \mathcal{A}(s)} \sum_{s'} p(s' \mid s, a) \left[ r(s, a, s') + \gamma V_k^{\pi_i}(s') \right]$$



**Notes**

A few demo runs of `mdp_agents.py`.

Note that the value is taken from "old policy" on RHS.

# Policy iteration algorithm

**function** POLICY-ITERATION(env) **returns:** policy $\pi$
  **input:** env - MDP problem
  $\pi(s) \leftarrow$ random $a \in A(s)$ in all states
  $V(s) \leftarrow 0$ in all states
  **repeat** $\triangleright$ iterate values until no change in policy
    $V \leftarrow$ POLICY-EVALUATION($\pi$, $V$, env)
    unchanged $\leftarrow$ True
    **for each** state $s$ **in** $S$ **do**
      **if** $\max\limits_{a \in A(s)} \sum_{s'} P(s'|a,s)V(s') > \sum_{s'} P(s'|s,\pi(s))V(s')$ **then**
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**Notes**

# Policy vs. Value iteration

- Value iteration.
  - Iteration updates values and policy. Although policy implicitly – extracted from values
  - No track of policy.
- Policy iteration.
  - Update utilities is fast – only one action per state.
  - New policy from values (slower)
  - New policy is better or done.
- Both methods belong to Dynamic programming realm.



**Notes**

Complexity (of one iteration step):
Value iteration: $O(S^2 * A)$
For every state (LHS), there can be up to $\sharp S$ also on RHS – if every other state was reachable from the current state.
In addition, all actions from every state need to be considered.
$Max(A)$ **does not** change often.
Policy often converges long before the values.
Policy evaluation: $O(S^3)$ (after AIMA, pg. 657)
The Bellman equations are *linear* because the max operator is gone.
For $\sharp S$ states, we have $\sharp S$ equations, which can be solved exactly in time $O(S^3)$ using standard linear algebra methods.
For small state spaces - ok.

For large state spaces - may be prohibitive $\rightarrow$ *modified policy iteration* with only a certain number of simplified Bellman update.

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# References

Further reading: Chapter 17 of [1] however, policy iteration is quite compact there. More detailed discussion can be found in chapter Dynamic programming in [2] with slightly different notation, though. This lecture has been also greatly inspired by the 9th lecture of CS 188 at http://ai.berkeley.edu as it convincingly motivates policy search and offers an alternative convergence proof of the value iteration method.

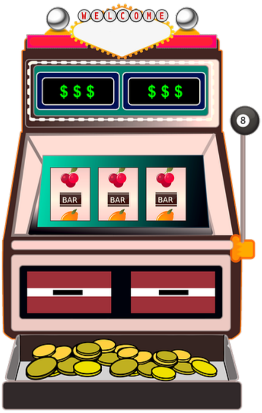

[1] Stuart Russell and Peter Norvig.
*Artificial Intelligence: A Modern Approach*.
Prentice Hall, 3rd edition, 2010.
http://aima.cs.berkeley.edu/.

[2] Richard S. Sutton and Andrew G. Barto.
*Reinforcement Learning; an Introduction*.
MIT Press, 2nd edition, 2018.
http://www.incompleteideas.net/book/the-book-2nd.html.




**Notes**

# Bandits







**Notes**