Fundamental matrix, properties

Suppose two cameras with camera projection matrices $\mathtt{P}_1$ and $\mathtt{P}_2$, i. e. there are three bases δ, β_1 and β_2 (first one is orthonormal basis from world coordinate system). There exist transformation matrices $\mathtt{A}_1$, $\mathtt{A}_2$ and camera centers C_1 and C_2 such that

$$\mathtt{P}_1 = [\mathtt{A}_1 \,|\, -\mathtt{A}_1 \vec{C}_{1\delta}], \quad \mathtt{P}_2 = [\mathtt{A}_2 \,|\, -\mathtt{A}_2 \vec{C}_{2\delta}]$$

Note that $\vec{x}_{\beta_i} = \mathtt{A}_i \vec{x}_\delta$ for arbitrary vector $\vec{x}$, so $\vec{x}_\delta = \mathtt{A}_i^{-1} \vec{x}_{\beta_i}$.

The *fundamental matrix* $\mathtt{F}$ is defined by

$$\mathtt{F} = \mathtt{A}_2^{-\top} ([C_{2\delta} - C_{1\delta}]_\times) \mathtt{A}_1^{-1}.$$

where $[\vec{u}]_\times \in \mathbb{R}^{3\times3}$ is a matrix which does "vector product calculation using matrix multiplication". More precisely, $([\vec{u}_\delta]_\times) \vec{v}_\delta$ gives coordinates of vector product $\vec{u} \times \vec{v}$ w.r.t. the basis δ. But there is an important condition: the basis δ must be othonormal. If $\vec{u}_\delta = [u_1, u_2, u_3]^\top$ and $\vec{v}_\delta = [v_1, v_2, v_3]^\top$ then

$$[\vec{u}_\delta]_\times = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ -u_1 v_3 + u_3 v_1 \\ u_1 v_2 - u_2 v_1 \end{bmatrix} = (\vec{u} \times \vec{v})_\delta$$

It is obvious that rank of $[\vec{u}_\delta]_\times$ is two for nonzero $\vec{u}$, so $\operatorname{rank} \mathtt{F} = 2$ for different camera centers.

Suppose the figure 12.2 from [1], where $\vec{x}_i$ are projections of a point X, $\vec{e}_i$ are eipoles, $\vec{l}_i$ are epipolar lines and XC_1C_2 is epipolar plane σ. We'll prove the following properties of the fundamental matrix $\mathtt{F}$:

$$(\vec{x}_{2\beta_2})^\top \mathtt{F}\, \vec{x}_{1\beta_1} = 0 \tag{1}$$
$$\mathtt{F}\, \vec{e}_{1\beta_1} = \vec{0} \tag{2}$$
$$\mathtt{F}^\top \vec{e}_{2\beta_2} = \vec{0} \tag{3}$$
$$\vec{l}_{1\bar{\beta}_1} = \mathtt{F}^\top \vec{x}_{2\beta_2} \tag{4}$$
$$\vec{l}_{2\bar{\beta}_2} = \mathtt{F}\, \vec{x}_{1\beta_1} \tag{5}$$

Ad (1):

$$\begin{aligned}(\vec{x}_{2\beta_2})^\top \mathtt{F}\, \vec{x}_{1\beta_1} &= (\vec{x}_{2\beta_2})^\top \mathtt{A}_2^{-\top} ([C_{2\delta} - C_{1\delta}]_\times) \mathtt{A}_1^{-1} \vec{x}_{1\beta_1} \\ &= (\mathtt{A}^{-1} \vec{x}_{2\beta_2})^\top ([C_{2\delta} - C_{1\delta}]_\times) \vec{x}_{1\delta} = (\vec{x}_{2\delta})^\top ((\vec{C}_2 - \vec{C}_1) \times \vec{x}_1)_\delta.\end{aligned}$$

Because the vector $((\vec{C}_1 - \vec{C}_1) \times \vec{x}_1)$ is perpendicular to the epipolar plane σ and the vector $\vec{x}_2$ is included in the epipolar plane σ, the dot product of these vectors must be zero. And the dot product of coordinates of these vectors w.r.t. orthonormal basis is zero too.

Ad (2):

$$\mathtt{F}\, \vec{e}_{1\beta_1} = \mathtt{A}_2^{-\top} ([C_{2\delta} - C_{1\delta}]_\times) \mathtt{A}_1^{-1} \vec{e}_{1\beta_1} = \mathtt{A}_2^{-\top} ([C_{2\delta} - C_{1\delta}]_\times) \vec{e}_{1\delta} = \mathtt{A}_2^{-\top} ((\vec{C}_2 - \vec{C}_1) \times \vec{e}_1)_\delta.$$

Because $\vec{e}_1$ is parallel with $\vec{C}_2 - \vec{C}_1$, the vector product must be zero vector. So, we have $\mathtt{A}_2^{-\top} \vec{0} = \vec{0}$.

Ad (3):

$$\mathtt{F}^\top \vec{e}_{2\beta_2} = \mathtt{A}_1^{-\top} ([C_{2\delta} - C_{1\delta}]_\times)^\top \mathtt{A}_2^{-1} \vec{e}_{2\beta_2} = -\mathtt{A}_1^{-\top} ([C_{2\delta} - C_{1\delta}]_\times) \vec{e}_{2\delta} = -\mathtt{A}_1^{-\top} ((\vec{C}_2 - \vec{C}_1) \times \vec{e}_2)_\delta.$$

Because $\vec{e}_2$ is parallel with $\vec{C}_2 - \vec{C}_1$, the vector product must be zero vector. So, we have $-\mathtt{A}_1^{-\top} \vec{0} = \vec{0}$.

Ad (4): We know that $(\vec{x}_{2\beta_2})^\top \mathtt{F}\, \vec{e}_{1\beta_1} = 0$ from (2) and $(\vec{x}_{2\beta_2})^\top \mathtt{F}\, \vec{x}_{1\beta_1} = 0$ from (1). These two equations say $\vec{z}^\top \vec{e}_{1\beta_1} = 0$ and $\vec{z}^\top \vec{x}_{1\beta_1} = 0$ when $\vec{z} = \mathtt{F}^\top \vec{x}_{2\beta_2}$. It means that $\vec{z}$ includes homogeneous coordinates of a line which goes through the points $\vec{e}_1$ and $\vec{x}_1$. But this is epipolar line, so $\vec{z} = \vec{l}_{1\bar{\beta}_1}$. The property (5) should be proven analogous.

[1] *Tomas Pajdla*: Elements of Geometry for Computer Vision (text available in materials of "Geometry of Computer Vision and Computer Graphics" Course)