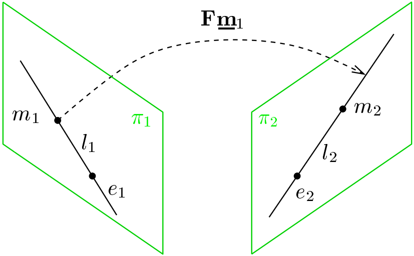
▶Some Mappings by the Fundamental Matrix



$$0 = \underline{\mathbf{m}}_2^\top \mathbf{F}\, \underline{\mathbf{m}}_1$$

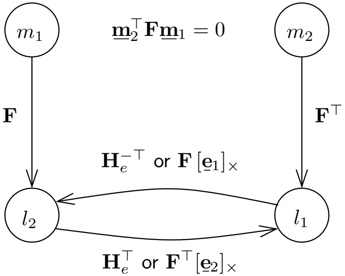
$$\underline{\mathbf{e}}_1 \simeq \mathrm{null}(\mathbf{F}), \qquad \underline{\mathbf{e}}_2 \simeq \mathrm{null}(\mathbf{F}^\top)$$

$$\underline{\mathbf{e}}_1 \simeq \mathbf{H}_e^{-1}\underline{\mathbf{e}}_2 \qquad \underline{\mathbf{e}}_2 \simeq \mathbf{H}_e\underline{\mathbf{e}}_1$$

$$\underline{\mathbf{l}}_1 \simeq \mathbf{F}^\top\underline{\mathbf{m}}_2 \qquad \underline{\mathbf{l}}_2 \simeq \mathbf{F}\underline{\mathbf{m}}_1$$

$$\underline{\mathbf{l}}_1 \simeq \mathbf{H}_e^\top\underline{\mathbf{l}}_2 \qquad \underline{\mathbf{l}}_2 \simeq \mathbf{H}_e^{-\top}\underline{\mathbf{l}}_1$$

$$\underline{\mathbf{l}}_1 \simeq \mathbf{F}^\top[\underline{\mathbf{e}}_2]_\times\underline{\mathbf{l}}_2 \qquad \underline{\mathbf{l}}_2 \simeq \mathbf{F}[\underline{\mathbf{e}}_1]_\times\underline{\mathbf{l}}_1$$



- $\mathbf{F}[\underline{\mathbf{e}}_1]_\times$ maps lines to lines but it is not a homography
- $\mathbf{H}_e = \mathbf{Q}_2\mathbf{Q}_1^{-1}$ is the epipolar homography→77
  $\mathbf{H}_e^{-\top}$ maps epipolar lines to epipolar lines, where

$$\mathbf{H}_e = \mathbf{Q}_2\mathbf{Q}_1^{-1} = \mathbf{K}_2\mathbf{R}_{21}\mathbf{K}_1^{-1}$$

you have seen this →59

▶Representation Theorem for Fundamental Matrices

Theorem: Every 3×3 matrix of rank 2 is a fundamental matrix.

Proof.

Converse: By the definition $\mathbf{F} = \mathbf{H}^{-\top}[\underline{\mathbf{e}}_1]_\times$ is a 3×3 matrix of rank 2.

Direct:

1. let $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$ be the SVD of a 3×3 matrix $\mathbf{A}$ of rank 2; then $\mathbf{D} = \operatorname{diag}(\lambda_1, \lambda_2, 0)$, $\lambda_1, \lambda_2 > 0$
2. we can write $\mathbf{D} = \mathbf{B}\mathbf{C}$, where $\mathbf{B} = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$, $\mathbf{C} = \operatorname{diag}(1, 1, 0)$, $\lambda_3 = 1$ (w.l.o.g.)
3. then $\mathbf{A} = \mathbf{U}\mathbf{B}\mathbf{C}\mathbf{V}^\top = \mathbf{U}\mathbf{B}\mathbf{C}\underbrace{\mathbf{W}\mathbf{W}^\top}_{\mathbf{I}}\mathbf{V}^\top$ with $\mathbf{W}$ rotation
4. we look for a rotation $\mathbf{W}$ that maps $\mathbf{C}$ to a skew-symmetric $\mathbf{S}$, i.e. $\mathbf{S} = \mathbf{C}\mathbf{W}$
5. then $\mathbf{W} = \begin{bmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $|\alpha| = 1$, and $\mathbf{S} = [\mathbf{s}]_\times$, $\mathbf{s} = (0, 0, 1)$
6. we can write

$$\mathbf{A} = \mathbf{U}\mathbf{B}[\mathbf{s}]_\times \mathbf{W}^\top \mathbf{V}^\top \overset{\circledast 1}{=} \cdots = \underbrace{\mathbf{U}\mathbf{B}(\mathbf{V}\mathbf{W})^\top}_{\mathbf{H}^{-\top}}[\mathbf{v}_3]_\times, \qquad \mathbf{v}_3 \text{ – 3rd column of } \mathbf{V} \tag{12}$$

7. $\mathbf{H}$ regular $\Rightarrow$ $\mathbf{A}$ does the job of a fundamental matrix, with epipole $\mathbf{v}_3$ and epipolar homography $\mathbf{H}$ □

- we also got a (non-unique: $\alpha = \pm 1$) decomposition formula for fundamental matrices
- it follows there is no constraint on $\mathbf{F}$ except the rank

▶Representation Theorem for Essential Matrices

Theorem

Let $\mathbf{E}$ *be a* 3×3 *matrix with SVD* $\mathbf{E} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$. *Then* $\mathbf{E}$ *is essential iff* $\mathbf{D} \simeq \operatorname{diag}(1, 1, 0)$.

Proof.

Direct:

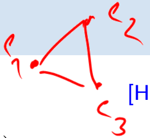
If $\mathbf{E}$ is an essential matrix, then the epipolar homography is a rotation (→77) and $\mathbf{U}\mathbf{B}(\mathbf{V}\mathbf{W})^\top$ in (12) must be orthogonal, therefore $\mathbf{B} = \lambda\mathbf{I}$.

Converse:

$\mathbf{E}$ is fundamental with $\mathbf{D} = \lambda \operatorname{diag}(1, 1, 0)$ then we do not need $\mathbf{B}$ (as if $\mathbf{B} = \lambda\mathbf{I}$) in (12) and $\mathbf{U}(\mathbf{V}\mathbf{W})^\top$ is orthogonal, as required.

□

▶Essential Matrix Decomposition



We are decomposing $\mathbf{E}$ to $\mathbf{E} = [-\mathbf{t}_{21}]_\times \mathbf{R}_{21} = \mathbf{R}_{21}[-\mathbf{R}_{21}^\top \mathbf{t}_{21}]_\times$ [H&Z, sec. 9.6]

1. compute SVD of $\mathbf{E} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$ and verify $\mathbf{D} = \lambda\,\mathrm{diag}(1,1,0)$
2. if $\det \mathbf{U} < 0$ change signs $\mathbf{U} \mapsto -\mathbf{U}$, $\mathbf{V} \mapsto -\mathbf{V}$ — the overall sign is dropped
3. compute

$$\mathbf{R}_{21} = \mathbf{U} \underbrace{\begin{bmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{W}} \mathbf{V}^\top, \quad \mathbf{t}_{21} = -\beta\,\mathbf{u}_3, \qquad |\alpha| = 1, \quad \beta \neq 0 \tag{13}$$

Notes

- $\mathbf{v}_3 \simeq \mathbf{R}_{21}^\top \mathbf{t}_{21}$ by (12), hence $\mathbf{R}_{21}\mathbf{v}_3 \simeq \mathbf{t}_{21} \simeq \mathbf{u}_3$ since it must fall in left null space by $\mathbf{E} \simeq [\mathbf{u}_3]_\times \mathbf{R}$
- $\mathbf{t}_{21}$ is recoverable up to scale β and direction $\mathrm{sign}\,\beta$
- the result for $\mathbf{R}_{21}$ is unique up to $\alpha = \pm 1$ — despite non-uniqueness of SVD
- change of sign in α rotates the solution by 180° about $\mathbf{t}_{21}$ — since $-\mathbf{W} = \mathbf{W}^\top$

  $\mathbf{R}(\alpha) = \mathbf{U}\mathbf{W}\mathbf{V}^\top$, $\mathbf{R}(-\alpha) = \mathbf{U}\mathbf{W}^\top\mathbf{V}^\top \Rightarrow \mathbf{T} = \mathbf{R}(-\alpha)\mathbf{R}^\top(\alpha) = \cdots = \mathbf{U}\,\mathrm{diag}(-1,-1,1)\mathbf{U}^\top$
  which is a rotation by 180° about $\mathbf{u}_3 = \mathbf{t}_{21}$:

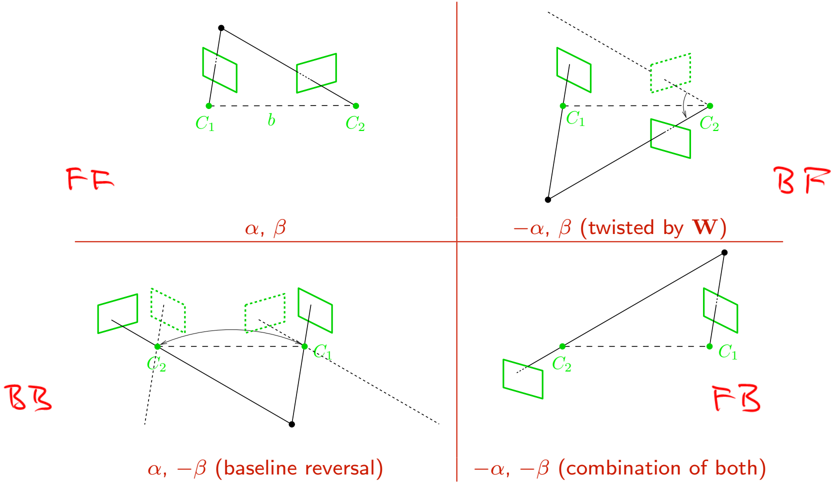
$U = [u_1, u_2, u_3]$

$$\mathbf{U}\,\mathrm{diag}(-1,-1,1)\mathbf{U}^\top \mathbf{u}_3 = \mathbf{U}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{u}_3$$

- 4 solution sets for 4 sign combinations of α, β — see next for geometric interpretation

▶Four Solutions to Essential Matrix Decomposition

Transform the world coordinate system so that the origin is in Camera 2. Then $\mathbf{t}_{21} = -\mathbf{b}$ and $\mathbf{W}$ rotates about the baseline $\mathbf{b}$. →76



- chirality constraint: all 3D points are in front of both cameras
- this singles-out the upper left case [H&Z, Sec. 9.6.3]

►7-Point Algorithm for Estimating Fundamental Matrix

Problem: Given a set $\{(x_i, y_i)\}_{i=1}^{k}$ of $k = 7$ correspondences, estimate f. m. $\mathbf{F}$.

$$\underline{\mathbf{y}}_i^\top \mathbf{F}\, \underline{\mathbf{x}}_i = 0, \quad i = 1, \ldots, k, \qquad \underline{\text{known}}: \quad \underline{\mathbf{x}}_i = (u_i^1, v_i^1, 1), \quad \underline{\mathbf{y}}_i = (u_i^2, v_i^2, 1)$$

terminology: correspondence = truth, later: match = algorithm's result; hypothesized corresp.

Solution:

$$\underline{\mathbf{y}}_i^\top \mathbf{F}\, \underline{\mathbf{x}}_i = (\underline{\mathbf{y}}_i \underline{\mathbf{x}}_i^\top) : \mathbf{F} = \left(\operatorname{vec}(\underline{\mathbf{y}}_i \underline{\mathbf{x}}_i^\top)\right)^\top \operatorname{vec}(\mathbf{F}),$$

$$\operatorname{vec}(\mathbf{F}) = \begin{bmatrix} f_{11} & f_{21} & f_{31} & \ldots & f_{33} \end{bmatrix}^\top \in \mathbb{R}^9$$ column vector from matrix

$$\mathbf{D} = \begin{bmatrix} \left(\operatorname{vec}(\mathbf{y}_1 \mathbf{x}_1^\top)\right)^\top \\ \left(\operatorname{vec}(\mathbf{y}_2 \mathbf{x}_2^\top)\right)^\top \\ \left(\operatorname{vec}(\mathbf{y}_3 \mathbf{x}_3^\top)\right)^\top \\ \vdots \\ \left(\operatorname{vec}(\mathbf{y}_k \mathbf{x}_k^\top)\right)^\top \end{bmatrix} = \begin{bmatrix} u_1^1 u_1^2 & u_1^1 v_1^2 & u_1^1 & u_1^2 v_1^1 & v_1^1 v_1^2 & v_1^1 & u_1^2 & v_1^2 & 1 \\ u_2^1 u_2^2 & u_2^1 v_2^2 & u_2^1 & u_2^2 v_2^1 & v_2^1 v_2^2 & v_2^1 & u_2^2 & v_2^2 & 1 \\ u_3^1 u_3^2 & u_3^1 v_3^2 & u_3^1 & u_3^2 v_3^1 & v_3^1 v_3^2 & v_3^1 & u_3^2 & v_3^2 & 1 \\ \vdots & & & & & & & & \vdots \\ u_k^1 u_k^2 & u_k^1 v_k^2 & u_k^1 & u_k^2 v_k^1 & v_k^1 v_k^2 & v_k^1 & u_k^2 & v_k^2 & 1 \end{bmatrix} \in \mathbb{R}^{k,9}$$

$$\mathbf{D}\operatorname{vec}(\mathbf{F}) = \mathbf{0}$$

▶7-Point Algorithm Continued

$$\mathbf{D}\,\mathrm{vec}(\mathbf{F}) = \mathbf{0}, \quad \mathbf{D} \in \mathbb{R}^{k,9}$$

- for $k = 7$ we have a rank-deficient system, the null-space of $\mathbf{D}$ is 2-dimensional
- but we know that $\det \mathbf{F} = 0$, hence
  1. find a basis of the null space of $\mathbf{D}$: $\mathbf{F}_1$, $\mathbf{F}_2$ — by SVD or QR factorization
  2. get up to 3 real solutions for α from

     $f(\alpha) = 0$ $\qquad \det(\alpha\mathbf{F}_1 + (1-\alpha)\mathbf{F}_2) = 0$ — cubic equation in α
  3. get up to 3 fundamental matrices $\mathbf{F} = \alpha_i\mathbf{F}_1 + (1-\alpha_i)\mathbf{F}_2$ — (check $\mathrm{rank}\,\mathbf{F} = 2$)

- the result may depend on image (domain) transformations
- normalization improves conditioning →91
- this gives a good starting point for the full algorithm →107
- dealing with mismatches need not be a part of the 7-point algorithm →108

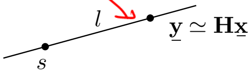
▶Degenerate Configurations for Fundamental Matrix Estimation

When is $\mathbf{F}$ not uniquely determined from any number of correspondences? [H&Z, Sec. 11.9]

1. when images are related by homography
   a) camera centers coincide $\mathbf{t}_{21} = 0$: $\quad \mathbf{H} = \mathbf{K}_2 \mathbf{R}_{21} \mathbf{K}_1^{-1}$
   b) camera moves but all 3D points lie in a plane $(\mathbf{n}, d)$: $\quad \mathbf{H} = \mathbf{K}_2 (\mathbf{R}_{21} - \mathbf{t}_{21} \mathbf{n}^\top / d) \mathbf{K}_1^{-1}$

   - in both cases: epipolar geometry is not defined
   - we do get a solution from the 7-point algorithm but it has the form of $\mathbf{F} = [\underline{\mathbf{s}}]_\times \mathbf{H}$

   note that $[\underline{\mathbf{s}}]_\times \mathbf{H} \simeq \mathbf{H}' [\underline{\mathbf{s}}']_\times$ →75



   - given (arbitrary) $\underline{\mathbf{s}}$
   - and correspondence $x \leftrightarrow y$
   - y is the image of x: $\underline{\mathbf{y}} \simeq \mathbf{H}\underline{\mathbf{x}}$
   - a necessary condition: $y \in l, \quad \underline{\mathbf{l}} \simeq \underline{\mathbf{s}} \times \mathbf{H}\underline{\mathbf{x}}$

$$0 = \underline{\mathbf{y}}^\top (\underline{\mathbf{s}} \times \mathbf{H}\underline{\mathbf{x}}) = \underline{\mathbf{y}}^\top [\underline{\mathbf{s}}]_\times \mathbf{H}\underline{\mathbf{x}} \quad \text{for any } \underline{\mathbf{x}}, \underline{\mathbf{s}} \text{ (!)}$$

2. both camera centers and all 3D points lie on a ruled quadric

   hyperboloid of one sheet, cones, cylinders, two planes

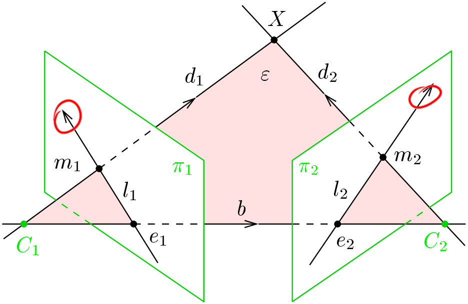
   - there are 3 solutions for $\mathbf{F}$

notes

- estimation of $\mathbf{E}$ can deal with planes: $[\underline{\mathbf{s}}]_\times \mathbf{H}$ is essential matrix iff $\underline{\mathbf{s}} = \lambda \mathbf{t}_{21}$ (see Case 1.b)
- a complete treatment with additional degenerate configurations in [H&Z, sec. 22.2]
- a stronger epipolar constraint could reject some configurations

A Note on Oriented Epipolar Constraint

- a tighter epipolar constraint preserves orientations
- requires all points and cameras be on the same side of the plane at infinity



$$\underline{\mathbf{e}}_2 \times \underline{\mathbf{m}}_2 \overset{+}{\simeq} \mathbf{F}\,\underline{\mathbf{m}}_1$$

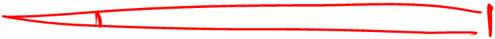
notation: $\underline{\mathbf{m}} \overset{+}{\simeq} \underline{\mathbf{n}}$ means $\underline{\mathbf{m}} = \lambda \underline{\mathbf{n}}$, $\lambda > 0$

- we can read the constraint as $\underline{\mathbf{e}}_2 \times \underline{\mathbf{m}}_2 \overset{+}{\simeq} \mathbf{H}_e^{-\top}\,(\mathbf{e}_1 \times \underline{\mathbf{m}}_1)$
- note that the constraint is not invariant to the change of either sign of $\underline{\mathbf{m}}_i$
- all 7 correspondence in 7-point alg. must have the same sign — see later
- this may help reject some wrong matches, see →108 — [Chum et al. 2004]
- an even more tight constraint: scene points in front of both cameras — expensive

  this is called chirality constraint

▶5-Point Algorithm for Relative Camera Orientation

Problem: Given $\{m_i, m'_i\}_{i=1}^{5}$ corresponding image points and calibration matrix $\mathbf{K}$, recover the camera motion $\mathbf{R}$, $\mathbf{t}$.



Obs:

1. $\mathbf{E}$ – 8 numbers
2. $\mathbf{R}$ – 3DOF, $\mathbf{t}$ – 2DOF only, in total 5 DOF $\rightarrow$ we need $8 - 5 = 3$ constraints on $\mathbf{E}$
3. $\mathbf{E}$ essential iff it has two equal singular values and the third is zero $\rightarrow$80

This gives an equation system:

$v_i^\top U D V v_i = 0$

$D = \lambda \operatorname{diag}(1,1,0)$

$$\underline{\mathbf{v}}_i^\top \mathbf{E}\, \underline{\mathbf{v}}'_i = 0 \qquad \text{5 linear constraints } (\underline{\mathbf{v}} \simeq \mathbf{K}^{-1}\underline{\mathbf{m}})$$

$$\det \mathbf{E} = 0 \qquad \text{1 cubic constraint}$$

$$\mathbf{E}\mathbf{E}^\top\mathbf{E} - \frac{1}{2}\operatorname{tr}(\mathbf{E}\mathbf{E}^\top)\mathbf{E} = \mathbf{0} \qquad \text{9 cubic constraints, 2 independent}$$

⊛ P1; 1pt: verify this equation from $\mathbf{E} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$, $\mathbf{D} = \lambda \operatorname{diag}(1, 1, 0)$

1. estimate $\mathbf{E}$ by SVD from $\underline{\mathbf{v}}_i^\top \mathbf{E}\, \underline{\mathbf{v}}'_i = 0$ by the null-space method — 4D null space
2. this gives $\mathbf{E} = x\mathbf{E}_1 + y\mathbf{E}_2 + z\mathbf{E}_3 + \mathbf{E}_4$
3. at most 10 (complex) solutions for x, y, z from the cubic constraints

- when all 3D points lie on a plane: at most 2 real solutions (twisted-pair) — can be disambiguated in 3 views or by chirality constraint (→82) unless all 3D points are closer to one camera
- 6-point problem for unknown f [Kukelova et al. BMVC 2008]
- resources at http://cmp.felk.cvut.cz/minimal/5_pt_relative.php

Thank You