# ▶The Triangulation Problem

**Problem:** Given cameras $\mathbf{P}_1$, $\mathbf{P}_2$ and a correspondence $x \leftrightarrow y$ compute a 3D point $\mathbf{X}$ projecting to $x$ and $y$

$$\lambda_1\,\underline{\mathbf{x}} = \mathbf{P}_1\underline{\mathbf{X}}, \qquad \lambda_2\,\underline{\mathbf{y}} = \mathbf{P}_2\underline{\mathbf{X}}, \qquad \underline{\mathbf{x}} = \begin{bmatrix} u^1 \\ v^1 \\ 1 \end{bmatrix}, \qquad \underline{\mathbf{y}} = \begin{bmatrix} u^2 \\ v^2 \\ 1 \end{bmatrix}, \qquad \mathbf{P}_i = \begin{bmatrix} (\mathbf{p}_1^i)^\top \\ (\mathbf{p}_2^i)^\top \\ (\mathbf{p}_3^i)^\top \end{bmatrix}$$

## Linear triangulation method

$$u^1\,(\mathbf{p}_3^1)^\top\underline{\mathbf{X}} = (\mathbf{p}_1^1)^\top\underline{\mathbf{X}}, \qquad u^2\,(\mathbf{p}_3^2)^\top\underline{\mathbf{X}} = (\mathbf{p}_1^2)^\top\underline{\mathbf{X}},$$

$$v^1\,(\mathbf{p}_3^1)^\top\underline{\mathbf{X}} = (\mathbf{p}_2^1)^\top\underline{\mathbf{X}}, \qquad v^2\,(\mathbf{p}_3^2)^\top\underline{\mathbf{X}} = (\mathbf{p}_2^2)^\top\underline{\mathbf{X}},$$

Gives

$$\mathbf{D}\underline{\mathbf{X}} = \mathbf{0}, \qquad \mathbf{D} = \begin{bmatrix} u^1\,(\mathbf{p}_3^1)^\top - (\mathbf{p}_1^1)^\top \\ v^1\,(\mathbf{p}_3^1)^\top - (\mathbf{p}_2^1)^\top \\ u^2\,(\mathbf{p}_3^2)^\top - (\mathbf{p}_1^2)^\top \\ v^2\,(\mathbf{p}_3^2)^\top - (\mathbf{p}_2^2)^\top \end{bmatrix}, \qquad \mathbf{D} \in \mathbb{R}^{4,4}, \quad \underline{\mathbf{X}} \in \mathbb{R}^4 \tag{14}$$

- back-projected rays will generally not intersect due to image error, see next
- using Jack-knife (→63) not recommended — sensitive to small error
- we will use SVD (→89)
- but the result will not be invariant to projective frame
  replacing $\mathbf{P}_1 \mapsto \mathbf{P}_1\mathbf{H}$, $\mathbf{P}_2 \mapsto \mathbf{P}_2\mathbf{H}$ does not always result in $\underline{\mathbf{X}} \mapsto \mathbf{H}^{-1}\underline{\mathbf{X}}$
- note the homogeneous form in (14) can represent points at infinity

# ►The Least-Squares Triangulation by SVD

- if $\mathbf{D}$ is full-rank we may minimize the algebraic least-squares error

$$\varepsilon^2(\underline{\mathbf{X}}) = \|\mathbf{D}\underline{\mathbf{X}}\|^2 \quad \text{s.t.} \quad \|\underline{\mathbf{X}}\| = 1, \qquad \underline{\mathbf{X}} \in \mathbb{R}^4$$

svd(Q) = USV^T

- let $\mathbf{D}_i$ be the $i$-th row of $\mathbf{D}$, then

$\|z\|^2 = z^T \cdot z$

$$\|\mathbf{D}\underline{\mathbf{X}}\|^2 = \sum_{i=1}^{4} (\mathbf{D}_i\,\underline{\mathbf{X}})^2 = \sum_{i=1}^{4} \underline{\mathbf{X}}^\top \mathbf{D}_i^\top \mathbf{D}_i\,\underline{\mathbf{X}} = \underline{\mathbf{X}}^\top \mathbf{Q}\,\underline{\mathbf{X}}, \text{ where } \mathbf{Q} = \sum_{i=1}^{4} \mathbf{D}_i^\top \mathbf{D}_i = \mathbf{D}^\top \mathbf{D} \in \mathbb{R}^{4,4}$$

- we write the SVD of $\mathbf{Q}$ as $\mathbf{Q} = \sum_{j=1}^{4} \sigma_j^2\,\mathbf{u}_j \mathbf{u}_j^\top$, in which [Golub & van Loan 2013, Sec. 2.5]

$$\sigma_1^2 \geq \cdots \geq \sigma_4^2 \geq 0 \quad \text{and} \quad \mathbf{u}_l^\top \mathbf{u}_m = \begin{cases} 0 & \text{if } l \neq m \\ 1 & \text{otherwise} \end{cases}$$

- then $\underline{\mathbf{X}} = \arg \underbrace{\min_{\mathbf{q}, \|\mathbf{q}\|=1} \mathbf{q}^\top \mathbf{Q}\,\mathbf{q}}_{\sigma_4^2} = \mathbf{u}_4$

**Proof** (by contradiction).

Let $\bar{\mathbf{q}} = \sum_{i=1}^{4} a_i \mathbf{u}_i$ s.t. $\sum_{i=1}^{4} a_i^2 = 1$, then $\|\bar{\mathbf{q}}\| = 1$, and

$$\bar{\mathbf{q}}^\top \mathbf{Q}\,\bar{\mathbf{q}} = \sum_{j=1}^{4} \sigma_j^2 (\bar{\mathbf{q}}^\top \mathbf{u}_j)(\mathbf{u}_j^\top \bar{\mathbf{q}}) = \sum_{j=1}^{4} \sigma_j^2\,(\mathbf{u}_j^\top \bar{\mathbf{q}})^2 = \cdots = \sum_{j=1}^{4} a_j^2 \sigma_j^2 \geq \overbrace{\left(\sum_{j=1}^{4} a_j^2\right)}^{1} \sigma_4^2 = \sigma_4^2$$

□

- if $\sigma_4 \ll \sigma_3$, there is a unique solution $\underline{\mathbf{X}} = \mathbf{u}_4$ with residual error $(\mathbf{D}\,\underline{\mathbf{X}})^2 = \sigma_4^2$
  the quality (conditioning) of the solution may be expressed as $q = \sigma_3/\sigma_4$ (greater is better)

Matlab code for the least-squares solver:

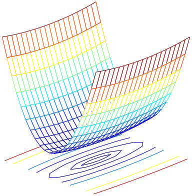
```
[U,O,V] = svd(D);   ← ?   works but we have to check!
X = V(:,end);
q = O(3,3)/O(4,4);
```

$$q = \underbrace{O(3,3)}_{\sigma_3^2}/\underbrace{O(4,4)}_{\sigma_4^2};$$

⊛ P1; 1pt: Why did we decompose $\mathbf{D}$ and not $\mathbf{Q} = \mathbf{D}^\top\mathbf{D}$?

# ▶Numerical Conditioning

- The equation $\mathbf{D}\underline{\mathbf{X}} = \mathbf{0}$ in (14) may be ill-conditioned for numerical computation, which results in a poor estimate for $\underline{\mathbf{X}}$.



**Why:** on a row of $\mathbf{D}$ there are big entries together with small entries, e.g. of orders    projection centers in mm, image points in px

$$\begin{bmatrix} 10^3 & 0 & 10^3 & 10^6 \\ 0 & 10^3 & 10^3 & 10^6 \\ 10^3 & 0 & 10^3 & 10^6 \\ 0 & 10^3 & 10^3 & 10^6 \end{bmatrix}$$

**Quick fix:**

1. re-scale the problem by a regular diagonal conditioning matrix $\mathbf{S} \in \mathbb{R}^{4,4}$

$$\mathbf{0} = \mathbf{D}\,\underline{\mathbf{X}} = \mathbf{D}\,\mathbf{S}\,\mathbf{S}^{-1}\underline{\mathbf{X}} = \bar{\mathbf{D}}\,\bar{\underline{\mathbf{X}}}$$

   choose $\mathbf{S}$ to make the entries in $\hat{\mathbf{D}}$ all smaller than unity in absolute value:

$$\mathbf{S} = \operatorname{diag}(10^{-3}, 10^{-3}, 10^{-3}, 10^{-6})$$

```
S = diag(1./max(abs(D), 1))
```

2. solve for $\bar{\underline{\mathbf{X}}}$ as before
3. get the final solution as $\underline{\mathbf{X}} = \mathbf{S}\,\bar{\underline{\mathbf{X}}}$

- when SVD is used in camera resection, conditioning is essential for success →62

# Algebraic Error vs Reprojection Error

- algebraic error ($c$ – camera index, $(u^c, v^c)$ – image coordinates) from SVD →90

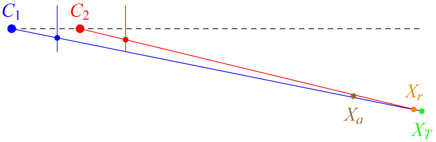
$$\varepsilon^2(\underline{\mathbf{X}}) = \sigma_4^2 = \sum_{c=1}^{2} \left[ \left( u^c (\mathbf{p}_3^c)^\top \underline{\mathbf{X}} - (\mathbf{p}_1^c)^\top \underline{\mathbf{X}} \right)^2 + \left( v^c (\mathbf{p}_3^c)^\top \underline{\mathbf{X}} - (\mathbf{p}_2^c)^\top \underline{\mathbf{X}} \right)^2 \right]$$

- reprojection error

$$e^2(\underline{\mathbf{X}}) = \sum_{c=1}^{2} \left[ \left( u^c - \frac{(\mathbf{p}_1^c)^\top \underline{\mathbf{X}}}{(\mathbf{p}_3^c)^\top \underline{\mathbf{X}}} \right)^2 + \left( v^c - \frac{(\mathbf{p}_2^c)^\top \underline{\mathbf{X}}}{(\mathbf{p}_3^c)^\top \underline{\mathbf{X}}} \right)^2 \right]$$

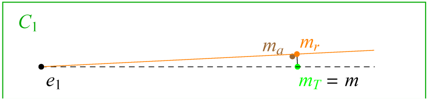
- algebraic error zero ⇔ reprojection error zero    $\sigma_4 = 0 \Rightarrow$ non-trivial null space
- epipolar constraint satisfied ⇒ equivalent results
- in general: minimizing algebraic error is cheap but it gives inferior results
- minimizing reprojection error is expensive but it gives good results
- the midpoint of the common perpendicular to both optical rays gives about 50% greater error in 3D
- the golden standard method – deferred to →104

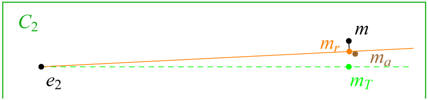
**Ex:**




- forward camera motion
- error $f/50$ in image 2, orthogonal to epipolar plane

$X_T$ – noiseless ground truth position
$X_r$ – reprojection error minimizer
$X_a$ – algebraic error minimizer
$m$ – measurement ($m_T$ with noise in $v^2$)



# ▶We Have Added to The ZOO

continuation from →68

| problem | given | unknown | slide |
|---|---|---|---|
| camera resection | 6 world–img correspondences $\left\{(X_i, m_i)\right\}_{i=1}^{6}$ | $\mathbf{P}$ | 62 |
| exterior orientation | $\mathbf{K}$, 3 world–img correspondences $\left\{(X_i, m_i)\right\}_{i=1}^{3}$ | $\mathbf{R}$, $\mathbf{t}$ | 66 |
| relative orientation | 3 world-world correspondences $\left\{(X_i, Y_i)\right\}_{i=1}^{3}$ | $\mathbf{R}$, $\mathbf{t}$ | 69 |
| fundamental matrix | 7 img–img correspondences $\left\{(m_i, m_i')\right\}_{i=1}^{7}$ | $\mathbf{F}$ | 83 |
| relative orientation | $\mathbf{K}$, 5 img–img correspondences $\left\{(m_i, m_i')\right\}_{i=1}^{5}$ | $\mathbf{R}$, $\mathbf{t}$ | 87 |
| triangulation | $\mathbf{P}_1$, $\mathbf{P}_2$, 1 img–img correspondence $(m_i, m_i')$ | $X$ | 88 |

A bigger ZOO at http://cmp.felk.cvut.cz/minimal/

**calibrated problems**

- have fewer degenerate configurations
- can do with fewer points (good for geometry proposal generators →117)

- algebraic error optimization (SVD) makes sense in camera resection and triangulation only
- but it is not the best method; we will now focus on 'optimizing optimally'

# Module V

## Optimization for 3D Vision



### covered by

[1] [H&Z] Secs: 11.4, 11.6, 4.7

[2] Fischler, M.A. and Bolles, R.C . Random Sample Consensus: A Paradigm for Model Fitting with Applications to Image Analysis and Automated Cartography. *Communications of the ACM* 24(6):381–395, 1981

### additional references

P. D. Sampson. Fitting conic sections to 'very scattered' data: An iterative refinement of the Bookstein algorithm. *Computer Vision, Graphics, and Image Processing*, 18:97–108, 1982.

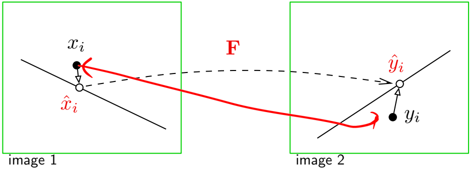
O. Chum, J. Matas, and J. Kittler. Locally optimized RANSAC. In *Proc DAGM*, LNCS 2781:236–243. Springer-Verlag, 2003.

O. Chum, T. Werner, and J. Matas. Epipolar geometry estimation via RANSAC benefits from the oriented epipolar constraint. In *Proc ICPR*, vol 1:112–115, 2004.

# ▶The Concept of Error for Epipolar Geometry

**Problem:** Given at least 8 matched points $x_i \leftrightarrow y_j$ in a general position, estimate the most likely (or most probable) fundamental matrix $\mathbf{F}$.

$$\mathbf{x}_i = (u_i^1,\, v_i^1), \quad \mathbf{y}_i = (u_i^2,\, v_i^2), \qquad i = 1, 2, \ldots, k, \quad k \geq 8$$




- detected points (measurements) $x_i$, $y_i$
- we introduce matches $\mathbf{Z}_i = (u_i^1, v_i^1, u_i^2, v_i^2) \in \mathbb{R}^4$; $\quad S = \{\mathbf{Z}_i\}_{i=1}^k$
- corrected points $\hat{x}_i$, $\hat{y}_i$; $\quad \hat{\mathbf{Z}}_i = (\hat{u}_i^1, \hat{v}_i^1, \hat{u}_i^2, \hat{v}_i^2)$; $\hat{S} = \{\hat{\mathbf{Z}}_i\}_{i=1}^k$ are correspondences
- correspondences satisfy the epipolar geometry exactly $\underline{\hat{\mathbf{y}}}_i^\top \mathbf{F}\, \underline{\hat{\mathbf{x}}}_i = 0$, $i = 1, \ldots, k$
- small correction is more probable
- let $\mathbf{e}_i(\cdot)$ be the 'reprojection error' (vector) per match $i$,

$$\mathbf{e}_i(x_i, y_i \mid \hat{x}_i, \hat{y}_i, \mathbf{F}) = \begin{bmatrix} \mathbf{x}_i - \hat{\mathbf{x}}_i \\ \mathbf{y}_i - \hat{\mathbf{y}}_i \end{bmatrix} = \mathbf{e}_i(\mathbf{Z}_i \mid \hat{\mathbf{Z}}_i, \mathbf{F}) = \mathbf{Z}_i - \hat{\mathbf{Z}}_i(\mathbf{F}) \tag{15}$$

$$\|\mathbf{e}_i(\cdot)\|^2 \stackrel{\text{def}}{=} \mathbf{e}_i^2(\cdot) = \|\mathbf{x}_i - \hat{\mathbf{x}}_i\|^2 + \|\mathbf{y}_i - \hat{\mathbf{y}}_i\|^2 = \|\mathbf{Z}_i - \hat{\mathbf{Z}}_i(\mathbf{F})\|^2$$

- the total reprojection error (of all data) then is

$$L(S \mid \hat{S}, \mathbf{F}) = \sum_{i=1}^{k} \mathbf{e}_i^2(x_i, y_i \mid \hat{x}_i, \hat{y}_i, \mathbf{F}) \quad = \sum_{i=1}^{k} \mathbf{e}_i^2(\mathbf{Z}_i \mid \hat{\mathbf{Z}}_i, \mathbf{F})$$

- and the optimization problem is

$$(\hat{S}^*, \mathbf{F}^*) = \arg \min_{\substack{\mathbf{F} \\ \operatorname{rank}\mathbf{F} = 2}} \; \min_{\substack{\hat{S} \\ \underline{\hat{\mathbf{y}}}_i^\top \mathbf{F}\, \underline{\hat{\mathbf{x}}}_i = 0}} \sum_{i=1}^{k} \mathbf{e}_i^2(x_i, y_i \mid \hat{x}_i, \hat{y}_i, \mathbf{F}) \tag{16}$$

## Three possible approaches

- they differ in how the correspondences $\hat{x}_i$, $\hat{y}_i$ are obtained:
  1. direct optimization of reprojection error over all variables $\hat{S}$, $\mathbf{F}$ →97
  2. Sampson optimal correction = partial correction of $\mathbf{Z}_i$ towards $\hat{\mathbf{Z}}_i$ used in an iterative minimization over $\mathbf{F}$ →98
  3. removing $\hat{x}_i$, $\hat{y}_i$ altogether = marginalization of $L(S, \hat{S} \mid \mathbf{F})$ over $\hat{S}$ followed by minimization over $\mathbf{F}$ not covered, the marginalization is difficult

# Method 1: Geometric Error Optimization

- we need to encode the constraints $\underline{\hat{\mathbf{y}}}_i \, \mathbf{F} \, \underline{\hat{\mathbf{x}}}_i = 0$, $\operatorname{rank} \mathbf{F} = 2$
- idea: reconstruct 3D point via equivalent projection matrices and use reprojection error
- equivalent projection matrices are see [H&Z,Sec. 9.5] for complete characterization

$$\mathbf{P}_1 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}, \quad \mathbf{P}_2 = \begin{bmatrix} [\underline{\mathbf{e}}_2]_\times \mathbf{F} + \underline{\mathbf{e}}_2 \underline{\mathbf{e}}_1^\top & \underline{\mathbf{e}}_2 \end{bmatrix} \tag{17}$$

⊛ H3; 2pt: Assuming $\underline{\mathbf{e}}_1$, $\underline{\mathbf{e}}_2$ are epipoles of $\mathbf{F}$, verify that $\mathbf{F}$ is a fundamental matrix of $\mathbf{P}_1$, $\mathbf{P}_2$.
Hint: $\mathbf{A}$ is skew symmetric iff $\mathbf{x}^\top \mathbf{A} \mathbf{x} = 0$ for all vectors $\mathbf{x}$.

---

1. compute $\mathbf{F}^{(0)}$ by the 7-point algorithm →83; construct camera $\mathbf{P}_2^{(0)}$ from $\mathbf{F}^{(0)}$ using (17)
2. triangulate 3D points $\hat{\mathbf{X}}_i^{(0)}$ from matches $(x_i, y_i)$ for all $i = 1, \ldots, k$ →88
3. starting from $\mathbf{P}_2^{(0)}$, $\hat{\mathbf{X}}^{(0)}$ minimize the reprojection error (15)

$$(\hat{\mathbf{X}}^*, \mathbf{P}_2^*) = \arg \min_{\mathbf{P}_2, \hat{\mathbf{X}}} \sum_{i=1}^{k} \mathbf{e}_i^2(\mathbf{Z}_i \mid \hat{\mathbf{Z}}_i(\hat{\mathbf{X}}_i, \mathbf{P}_2))$$

   where

$$\hat{\mathbf{Z}}_i = (\hat{\mathbf{x}}_i, \hat{\mathbf{y}}_i) \ \text{(Cartesian)}, \quad \underline{\hat{\mathbf{x}}}_i \simeq \mathbf{P}_1 \underline{\hat{\mathbf{X}}}_i, \ \ \underline{\hat{\mathbf{y}}}_i \simeq \mathbf{P}_2 \underline{\hat{\mathbf{X}}}_i \ \text{(homogeneous)}$$

   Non-linear, non-convex problem

4. compute $\mathbf{F}$ from $\mathbf{P}_1$, $\mathbf{P}_2^*$

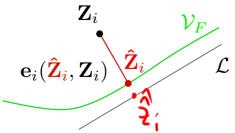
- $3k + 12$ parameters to be found: latent: $\hat{\mathbf{X}}_i$, for all $i$ (correspondences!), non-latent: $\mathbf{P}_2$
- minimal representation: $3k + 7$ parameters, $\mathbf{P}_2 = \mathbf{P}_2(\mathbf{F})$ →143
- there are pitfalls; this is essentially bundle adjustment; we will return to this later →134

# ▶Method 2: First-Order Error Approximation

**An elegant method for solving problems like (16):**

- we will get rid of the latent parameters $\hat{X}$ needed for obtaining the correction
  [H&Z, p. 287], [Sampson 1982]
- we will recycle the algebraic error $\boldsymbol{\varepsilon} = \underline{\mathbf{y}}^\top \mathbf{F}\, \underline{\mathbf{x}}$ from →83

- consider matches $\mathbf{Z}_i$, correspondences $\hat{\mathbf{Z}}_i$, and reprojection error $\mathbf{e}_i = \|\mathbf{Z}_i - \hat{\mathbf{Z}}_i\|^2$
- correspondences satisfy $\hat{\underline{\mathbf{y}}}_i^\top \mathbf{F}\, \hat{\underline{\mathbf{x}}}_i = 0$, $\qquad \hat{\underline{\mathbf{x}}}_i = (\hat{u}^1, \hat{v}^1, 1)$, $\hat{\underline{\mathbf{y}}}_i = (\hat{u}^2, \hat{v}^2, 1)$
- this is a manifold $\mathcal{V}_F \in \mathbb{R}^4$: a set of points $\hat{\mathbf{Z}} = (\hat{u}^1,\, \hat{v}^1,\, \hat{u}^2,\, \hat{v}^2)$ consistent with $\mathbf{F}$
- algebraic error vanishes for $\hat{\mathbf{Z}}_i$: $\boxed{\mathbf{0} = \boldsymbol{\varepsilon}_i(\hat{\mathbf{Z}}_i) = \hat{\underline{\mathbf{y}}}_i^\top \mathbf{F}\, \hat{\underline{\mathbf{x}}}_i}$




**Sampson's idea:** Linearize the algebraic error $\boldsymbol{\varepsilon}(\mathbf{Z})$ at $\mathbf{Z}_i$ (where it is non-zero) and evaluate the resulting linear function at $\hat{\mathbf{Z}}_i$ (where it is zero). The zero-crossing replaces $\mathcal{V}_F$ by a linear manifold $\mathcal{L}$. The point on $\mathcal{V}_F$ closest to $\mathbf{Z}_i$ is replaced by the closest point on $\mathcal{L}$.

$\cancel{e = -J^{-1}\varepsilon}$

$$\boldsymbol{\varepsilon}_i(\hat{\mathbf{Z}}_i) \;\approx\; \boldsymbol{\varepsilon}_i(\mathbf{Z}_i) + \underbrace{\frac{\partial \boldsymbol{\varepsilon}_i(\mathbf{Z}_i)}{\partial \mathbf{Z}_i}}_{J}\, \underbrace{(\hat{\mathbf{Z}}_i - \mathbf{Z}_i)}_{e} = 0$$

# ▶Sampson's Approximation of Reprojection Error

- linearize $\boldsymbol{\varepsilon}(\mathbf{Z})$ at match $\mathbf{Z}_i$, evaluate it at correspondence $\hat{\mathbf{Z}}_i$

$$0 = \boldsymbol{\varepsilon}_i(\hat{\mathbf{Z}}_i) \approx \boldsymbol{\varepsilon}_i(\mathbf{Z}_i) + \underbrace{\frac{\partial \boldsymbol{\varepsilon}_i(\mathbf{Z}_i)}{\partial \mathbf{Z}_i}}_{\mathbf{J}_i(\mathbf{Z}_i)} \underbrace{(\hat{\mathbf{Z}}_i - \mathbf{Z}_i)}_{\mathbf{e}_i(\hat{\mathbf{Z}}_i, \mathbf{Z}_i)} \stackrel{\text{def}}{=} \boldsymbol{\varepsilon}_i(\mathbf{Z}_i) + \mathbf{J}_i(\mathbf{Z}_i)\, \mathbf{e}_i(\hat{\mathbf{Z}}_i, \mathbf{Z}_i)$$

- goal: compute function $\mathbf{e}_i(\cdot)$ from $\boldsymbol{\varepsilon}_i(\cdot)$, where $\mathbf{e}_i(\cdot)$ is the distance of $\hat{\mathbf{Z}}_i$ from $\mathbf{Z}_i$
- we have a linear underconstrained equation for $\mathbf{e}_i(\cdot)$
- we look for a minimal $\mathbf{e}_i(\cdot)$ per match $i$

$$\mathbf{e}_i(\cdot)^* = \arg\min_{\mathbf{e}_i(\cdot)} \|\mathbf{e}_i(\cdot)\|^2 \quad \text{subject to} \quad \boldsymbol{\varepsilon}_i(\cdot) + \mathbf{J}_i(\cdot)\, \mathbf{e}_i(\cdot) = 0$$

- which has a closed-form solution note that $\mathbf{J}_i(\cdot)$ is not invertible! ⊛ P1; 1pt: derive $\mathbf{e}_i^*(\cdot)$

$$\begin{aligned} \mathbf{e}_i^*(\cdot) &= -\mathbf{J}_i^\top (\mathbf{J}_i \mathbf{J}_i^\top)^{-1} \boldsymbol{\varepsilon}_i(\cdot) \\ \|\mathbf{e}_i^*(\cdot)\|^2 &= \boldsymbol{\varepsilon}_i^\top(\cdot) (\mathbf{J}_i \mathbf{J}_i^\top)^{-1} \boldsymbol{\varepsilon}_i(\cdot) \end{aligned} \tag{18}$$

- this maps $\boldsymbol{\varepsilon}_i(\cdot)$ to an estimate of $\mathbf{e}_i(\cdot)$ per correspondence
- we often do not need $\mathbf{e}_i$, just $\|\mathbf{e}_i\|^2$ exception: triangulation →104
- the unknown parameters $\mathbf{F}$ are inside: $\mathbf{e}_i = \mathbf{e}_i(\mathbf{F})$, $\boldsymbol{\varepsilon}_i = \boldsymbol{\varepsilon}_i(\mathbf{F})$, $\mathbf{J}_i = \mathbf{J}_i(\mathbf{F})$

# ▶Example: Fitting A Circle To Scattered Points

**Problem:** Fit a zero-centered circle $\mathcal{C}$ to a set of 2D points $\{x_i\}_{i=1}^{k}$, $\mathcal{C}\colon\ \|\mathbf{x}\|^2 - r^2 = 0$.

1. consider radial error as the 'algebraic error' $\boldsymbol{\varepsilon}(\mathbf{x}) = \|\mathbf{x}\|^2 - r^2$
2. linearize it at $\hat{\mathbf{x}}$ — we are dropping $i$ in $\boldsymbol{\varepsilon}_i$, $\mathbf{e}_i$ etc for clarity

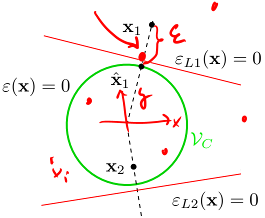
$$\boldsymbol{\varepsilon}(\hat{\mathbf{x}}) \approx \boldsymbol{\varepsilon}(\mathbf{x}) + \underbrace{\frac{\partial \boldsymbol{\varepsilon}(\mathbf{x})}{\partial \mathbf{x}}}_{\mathbf{J}(\mathbf{x}) = 2\mathbf{x}^\top} \underbrace{(\hat{\mathbf{x}} - \mathbf{x})}_{\mathbf{e}(\hat{\mathbf{x}},\mathbf{x})} = \cdots = 2\,\mathbf{x}^\top \hat{\mathbf{x}} - (r^2 + \|\mathbf{x}\|^2) \overset{\text{def}}{=} \boldsymbol{\varepsilon}_L(\hat{\mathbf{x}}) = 0$$

$\boldsymbol{\varepsilon}_L(\hat{\mathbf{x}}) = 0$ is a line with normal $\frac{\mathbf{x}}{\|\mathbf{x}\|}$ and intercept $\frac{r^2 + \|\mathbf{x}\|^2}{2\|\mathbf{x}\|}$ — not tangent to $\mathcal{C}$, outside!

3. using (18), express error approximation $\mathbf{e}^*$ as

$$\|\mathbf{e}^*\|^2 = \boldsymbol{\varepsilon}^\top (\mathbf{J}\mathbf{J}^\top)^{-1} \boldsymbol{\varepsilon} = \frac{(\|\mathbf{x}\|^2 - r^2)^2}{4\|\mathbf{x}\|^2}$$

4. fit circle



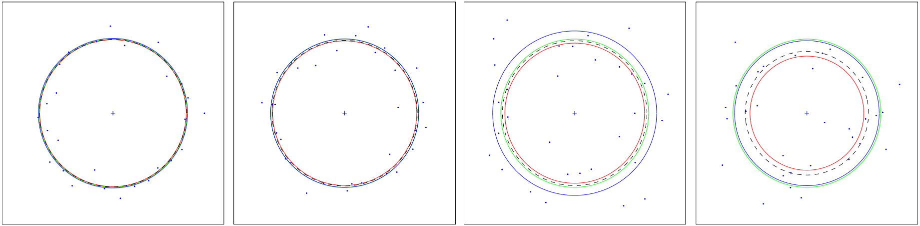

$$r^* = \arg\min_r \sum_{i=1}^{k} \frac{(\|\mathbf{x}_i\|^2 - r^2)^2}{4\|\mathbf{x}_i\|^2} = \cdots = \left( \frac{1}{k} \sum_{i=1}^{k} \frac{1}{\|\mathbf{x}_i\|^2} \right)^{-\frac{1}{2}}$$

- this example results in a convex quadratic optimization problem
- note that

$$\arg\min_r \sum_{i=1}^{k} (\|\mathbf{x}_i\|^2 - r^2)^2 = \left( \frac{1}{k} \sum_{i=1}^{k} \|\mathbf{x}_i\|^2 \right)^{\frac{1}{2}}$$

# Circle Fitting: Some Results

medium radial noise | medium isotropic noise | big radial noise | big isotropic noise



opt: 1.8, Smp: <u>1.9</u>, dir: 2.3 | 1.8, 2.0, 2.2 | 1.6, <u>1.8</u>, 2.6 | 1.6, 2.0, 2.4

mean ranks over 10 000 random trials with $k = 32$ samples

green – ground truth

red – Sampson error minimizer

blue – direct radial error minimizer

black – optimal estimator for isotropic error

optimal estimator for isotropic error (black, dashed):

$$r \approx \frac{3}{4k} \sum_{i=1}^{k} \|\mathbf{x}_i\| + \sqrt{\left(\frac{3}{4k} \sum_{i=1}^{k} \|\mathbf{x}_i\|\right)^2 - \frac{1}{2k} \sum_{i=1}^{k} \|\mathbf{x}_i\|^2}$$

**which method is better?**

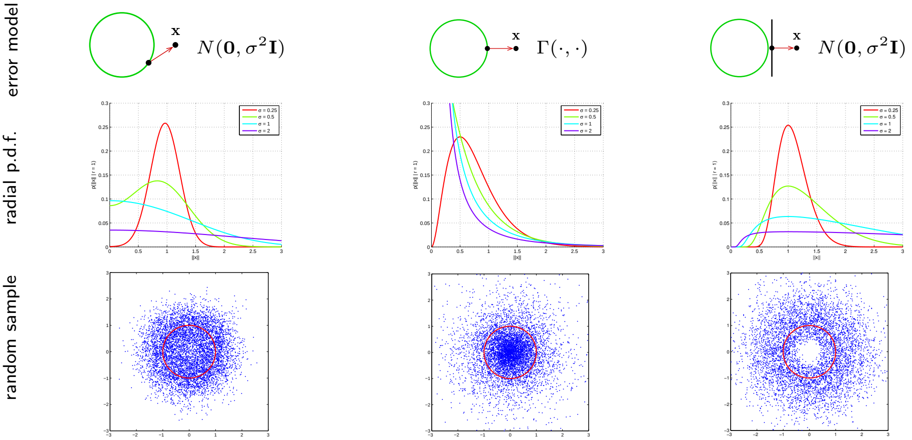
- error should model noise, radial noise and isotropic noise behave differently
- ground truth: Normally distributed isotropic error, Gamma-distributed radial error
- Sampson: <u>better</u> for the radial distribution model; Direct: better for the isotropic model
- no matter how corrected, the algebraic error minimizer is not an unbiased parameter estimator

Cramér-Rao bound tells us how close one can get with unbiased estimator and given $k$

# Discussion: On The Art of Probabilistic Model Design...

- a few models for fitting zero-centered circle $C$ of radius $r$ to points in $\mathbb{R}^2$

| marginalized over $C$ | orthogonal deviation from $C$ | Sampson approximation |
|---|---|---|




$p(\mathbf{x} \mid r)$

$$\approx \frac{1}{\sigma\sqrt{(2\pi)^3 r\,\|\mathbf{x}\|}}\, e^{-\frac{(\|\mathbf{x}\|-r)^2}{2\sigma^2}}$$

$$\frac{1}{2\pi\Gamma(\frac{r^2}{\sigma})}\,\frac{1}{\|\mathbf{x}\|^2}\left(\frac{r\|\mathbf{x}\|}{\sigma}\right)^{\frac{r^2}{\sigma}} e^{-\frac{r\|\mathbf{x}\|}{\sigma}}$$

$$\frac{1}{r\sigma\sqrt{(2\pi)^3}}\, e^{-\frac{e^2(\mathbf{x};r)}{2\sigma^2}}$$

- mode inside the circle
- models the inside well
- tends to normal distrib.

- peak at the center
- unusable for small radii
- tends to Dirac distrib.

- mode at the circle
- hole at the center
- tends to normal distrib.

# ▶Sampson Error for Fundamental Matrix Manifold

The epipolar algebraic error is

$$\varepsilon_i(\mathbf{F}) = \underline{\mathbf{y}}_i^\top \mathbf{F}\, \underline{\mathbf{x}}_i, \quad \mathbf{x}_i = (u_i^1,\, v_i^1), \quad \mathbf{y}_i = (u_i^2,\, v_i^2), \qquad \varepsilon_i \in \mathbb{R}$$

Let $\mathbf{F} = \begin{bmatrix} \mathbf{F}_1 & \mathbf{F}_2 & \mathbf{F}_3 \end{bmatrix}$ (per columns) $= \begin{bmatrix} (\mathbf{F}^1)^\top \\ (\mathbf{F}^2)^\top \\ (\mathbf{F}^3)^\top \end{bmatrix}$ (per rows), $\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then

**Sampson**

$$\mathbf{J}_i(\mathbf{F}) = \left[\frac{\partial \varepsilon_i(\mathbf{F})}{\partial u_i^1}, \frac{\partial \varepsilon_i(\mathbf{F})}{\partial v_i^1}, \frac{\partial \varepsilon_i(\mathbf{F})}{\partial u_i^2}, \frac{\partial \varepsilon_i(\mathbf{F})}{\partial v_i^2}\right] \qquad \mathbf{J}_i \in \mathbb{R}^{1,4}$$

derivatives over point coords.

$$= \left[(\mathbf{F}_1)^\top \underline{\mathbf{y}}_i,\ (\mathbf{F}_2)^\top \underline{\mathbf{y}}_i,\ (\mathbf{F}^1)^\top \underline{\mathbf{x}}_i,\ (\mathbf{F}^2)^\top \underline{\mathbf{x}}_i\right]$$

$$\mathbf{e}_i(\mathbf{F}) = -\frac{\mathbf{J}_i(\mathbf{F})\, \varepsilon_i(\mathbf{F})}{\|\mathbf{J}_i(\mathbf{F})\|^2} \qquad \mathbf{e}_i(\mathbf{F}) \in \mathbb{R}^4$$

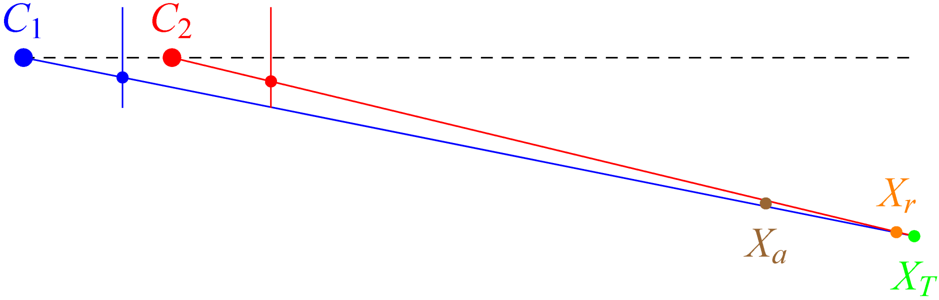
Sampson error vector

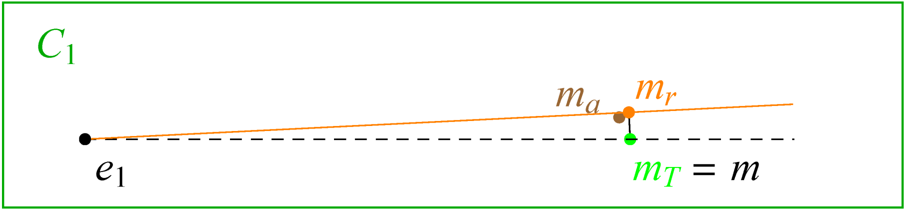
$$e_i(\mathbf{F}) = \|\mathbf{e}_i(\mathbf{F})\| = \frac{\varepsilon_i(\mathbf{F})}{\|\mathbf{J}_i(\mathbf{F})\|} = \frac{\underline{\mathbf{y}}_i^\top \mathbf{F} \underline{\mathbf{x}}_i}{\sqrt{\|\mathbf{S}\mathbf{F}\underline{\mathbf{x}}_i\|^2 + \|\mathbf{S}\mathbf{F}^\top \underline{\mathbf{y}}_i\|^2}} \qquad e_i(\mathbf{F}) \in \mathbb{R}$$

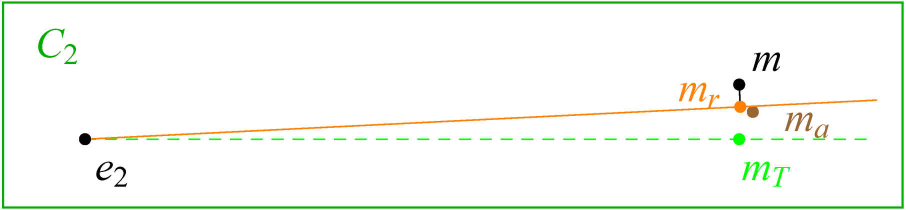
scalar Sampson error

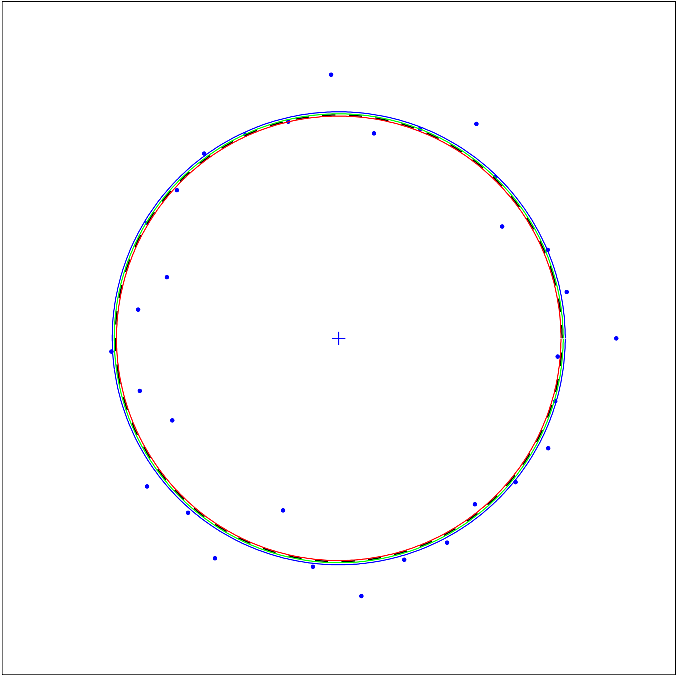
- Sampson error 'normalizes' the algebraic error
- automatically copes with multiplicative factors $\mathbf{F} \mapsto \lambda \mathbf{F}$
- actual optimization not yet covered →108

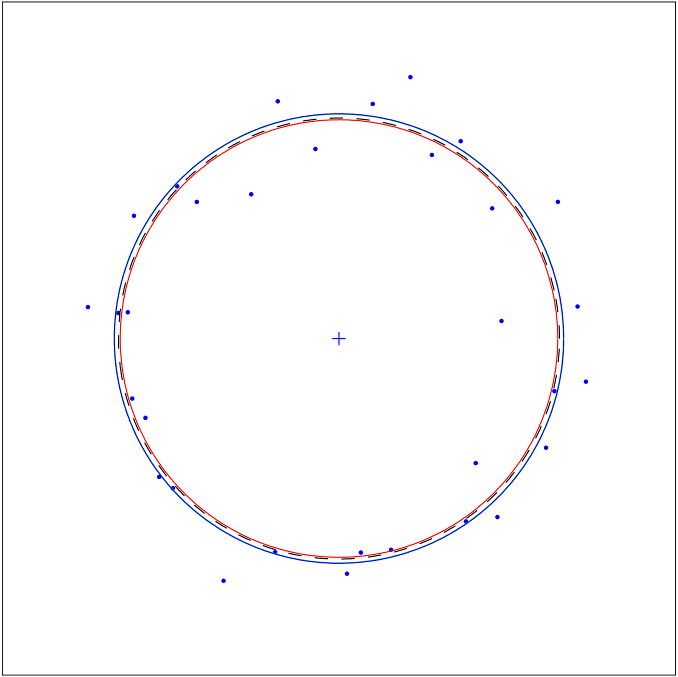
Thank You

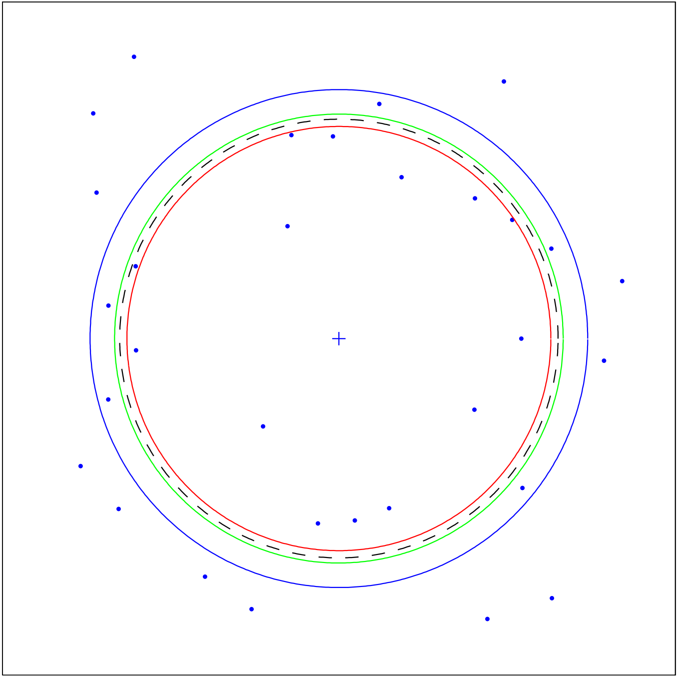

$C_1$
$C_2$
$X_r$
$X_a$
$X_T$

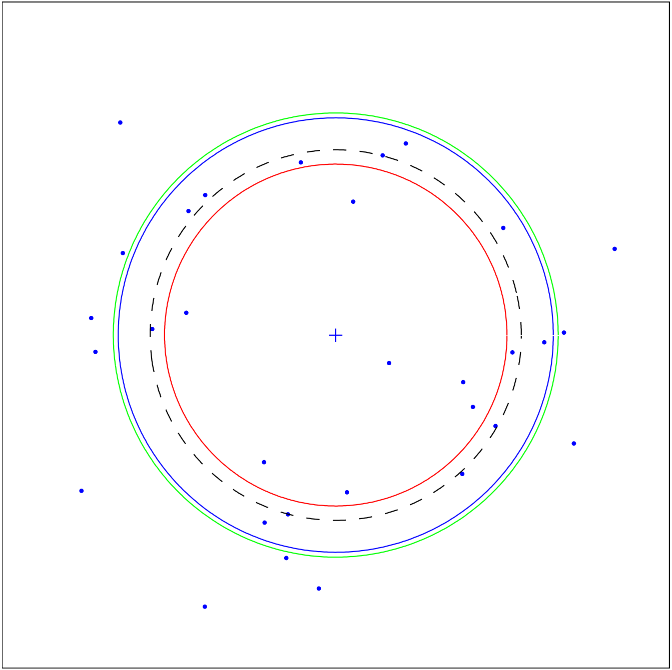

$C_1$
$m_a$
$m_r$
$e_1$
$m_T = m$

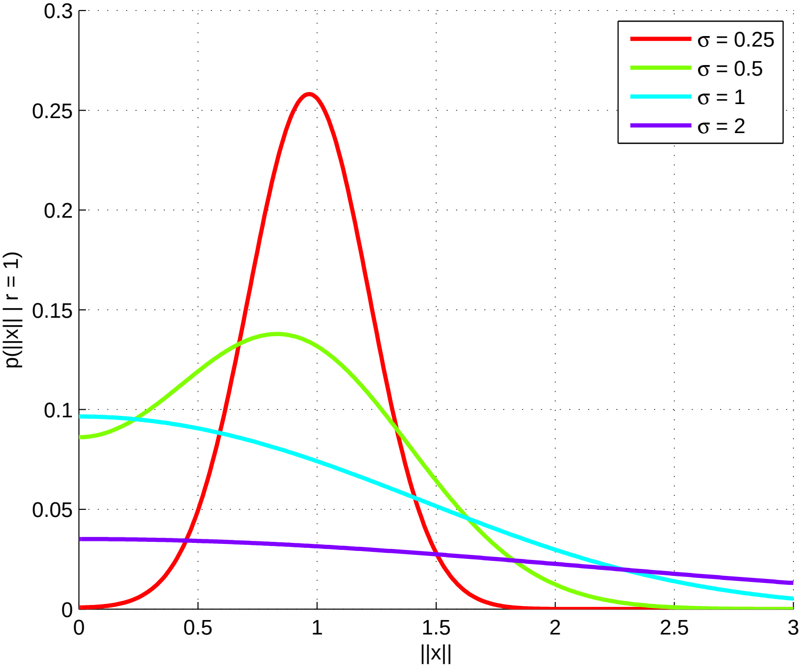

$C_2$
$m$
$m_r$
$m_a$
$e_2$
$m_T$







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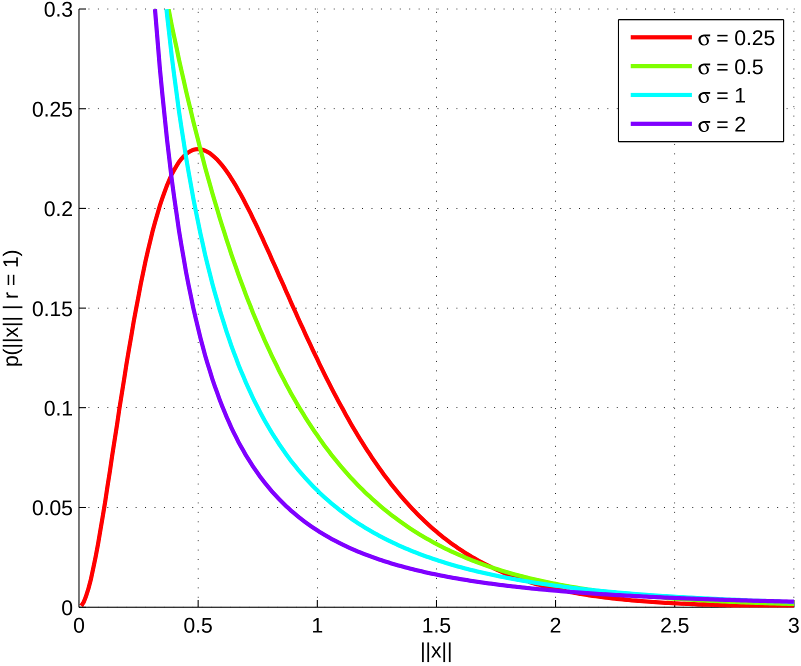
σ = 0.5
σ = 1
σ = 2
0.3
0.25
0.2
0.15
0.1
0.05
0
p(||x|| | r = 1)
0
0.5
1
1.5
2
2.5
3
||x||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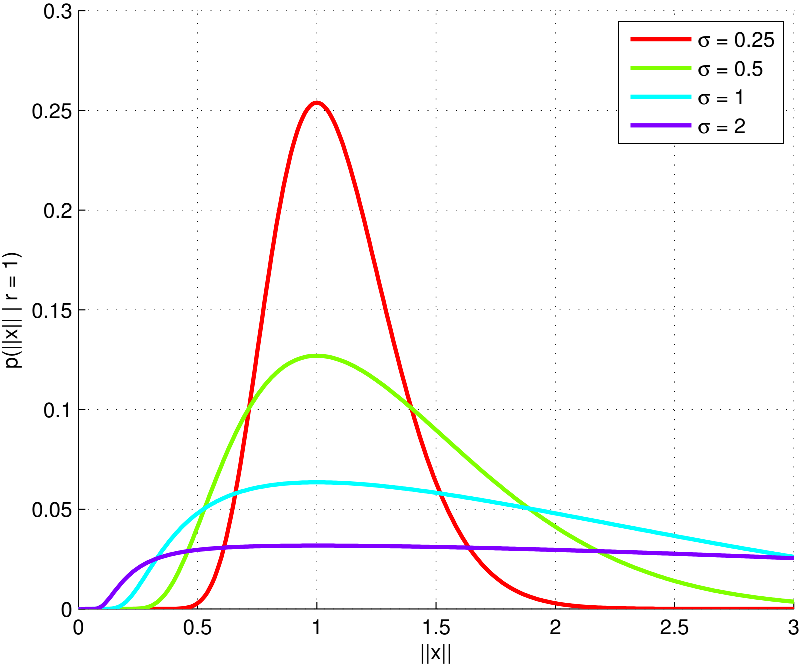
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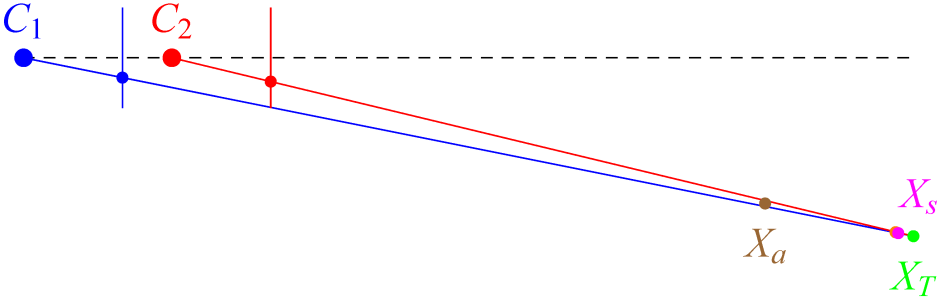
$C_1$
$C_2$
$X_s$
$X_a$
$X_T$