Method 1: Geometric Error Optimization

- we need to encode the constraints $\hat{\underline{\mathbf{y}}}_i \mathbf{F} \hat{\underline{\mathbf{x}}}_i = 0$, $\operatorname{rank} \mathbf{F} = 2$
- idea: reconstruct 3D point via equivalent projection matrices and use reprojection error
- equivalent projection matrices are see [H&Z,Sec. 9.5] for complete characterization

$$\mathbf{P}_1 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}, \quad \mathbf{P}_2 = \begin{bmatrix} [\underline{\mathbf{e}}_2]_\times \mathbf{F} + \underline{\mathbf{e}}_2 \underline{\mathbf{e}}_1^\top & \underline{\mathbf{e}}_2 \end{bmatrix} \tag{17}$$

⊛ H3; 2pt: Assuming $\underline{\mathbf{e}}_1$, $\underline{\mathbf{e}}_2$ are epipoles of $\mathbf{F}$, verify that $\mathbf{F}$ is a fundamental matrix of $\mathbf{P}_1$, $\mathbf{P}_2$.
Hint: $\mathbf{A}$ is skew symmetric iff $\mathbf{x}^\top \mathbf{A} \mathbf{x} = 0$ for all vectors $\mathbf{x}$.

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1. compute $\mathbf{F}^{(0)}$ by the 7-point algorithm →83; construct camera $\mathbf{P}_2^{(0)}$ from $\mathbf{F}^{(0)}$ using (17)
2. triangulate 3D points $\hat{\mathbf{X}}_i^{(0)}$ from matches (x_i, y_i) for all $i = 1, \ldots, k$ →88
3. starting from $\mathbf{P}_2^{(0)}$, $\hat{\mathbf{X}}^{(0)}$ minimize the reprojection error (15)

$$(\hat{\mathbf{X}}^*, \mathbf{P}_2^*) = \arg \min_{\mathbf{P}_2, \hat{\mathbf{X}}} \sum_{i=1}^{k} \mathbf{e}_i^2(\mathbf{Z}_i \mid \hat{\mathbf{Z}}_i(\hat{\mathbf{X}}_i, \mathbf{P}_2))$$

   where

$$\hat{\mathbf{Z}}_i = (\hat{\mathbf{x}}_i, \hat{\mathbf{y}}_i) \text{ (Cartesian)}, \quad \hat{\underline{\mathbf{x}}}_i \simeq \mathbf{P}_1 \hat{\underline{\mathbf{X}}}_i, \quad \hat{\underline{\mathbf{y}}}_i \simeq \mathbf{P}_2 \hat{\underline{\mathbf{X}}}_i \text{ (homogeneous)}$$

   Non-linear, non-convex problem
4. compute $\mathbf{F}$ from $\mathbf{P}_1$, $\mathbf{P}_2^*$

- $3k + 12$ parameters to be found: latent: $\hat{\mathbf{X}}_i$, for all i (correspondences!), non-latent: $\mathbf{P}_2$
- minimal representation: $3k + 7$ parameters, $\mathbf{P}_2 = \mathbf{P}_2(\mathbf{F})$ →143
- there are pitfalls; this is essentially bundle adjustment; we will return to this later →134

▸Back to Triangulation: The Golden Standard Method

Given $\mathbf{P}_1$, $\mathbf{P}_2$ and a correspondence $x \leftrightarrow y$, look for 3D point $\mathbf{X}$ projecting to x and y. →88

Idea:

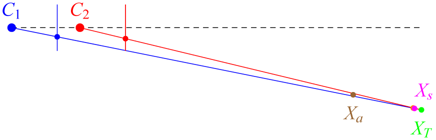
1. if not given, compute $\mathbf{F}$ from $\mathbf{P}_1$, $\mathbf{P}_2$, e.g. $\mathbf{F} = (\mathbf{Q}_1\mathbf{Q}_2^{-1})^\top[\mathbf{q}_1 - (\mathbf{Q}_1\mathbf{Q}_2^{-1})\mathbf{q}_2]_\times$
2. correct measurement by the linear estimate of the correction vector →99

$$\begin{bmatrix}\hat{u}^1\\ \hat{v}^1\\ \hat{u}^2\\ \hat{v}^2\end{bmatrix} \approx \begin{bmatrix}u^1\\ v^1\\ u^2\\ v^2\end{bmatrix} - \frac{\varepsilon}{\|\mathbf{J}\|^2}\mathbf{J}^\top = \begin{bmatrix}u^1\\ v^1\\ u^2\\ v^2\end{bmatrix} - \frac{\underline{\mathbf{y}}^\top\mathbf{F}\underline{\mathbf{x}}}{\|\mathbf{S}\mathbf{F}\underline{\mathbf{x}}\|^2 + \|\mathbf{S}\mathbf{F}^\top\underline{\mathbf{y}}\|^2}\begin{bmatrix}(\mathbf{F}_1)^\top\underline{\mathbf{y}}\\ (\mathbf{F}_2)^\top\underline{\mathbf{y}}\\ (\mathbf{F}^1)^\top\underline{\mathbf{x}}\\ (\mathbf{F}^2)^\top\underline{\mathbf{x}}\end{bmatrix}$$

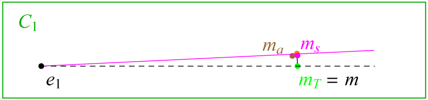
3. use the SVD triangulation algorithm with numerical conditioning →89; iteration possible

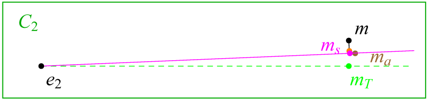
Ex (cont'd from →92):

$S = \mathrm{diag}(1,1,0)$



X_T – noiseless ground truth position
● – reprojection error minimizer
X_s – Sampson-corrected algebraic error minimizer
X_a – algebraic error minimizer
m – measurement (m_T with noise in v^2)





▶Back to Fundamental Matrix Estimation

Goal: Given a set $X = \{(x_i, y_i)\}_{i=1}^{k}$ of $k \gg 7$ inlier correspondences, compute a statistically efficient estimate for fundamental matrix $\mathbf{F}$.

What we have so far

- 7-point algorithm for $\mathbf{F}$ (5-point algorithm for $\mathbf{E}$) →83
- definition of Sampson error per correspondence $e_i(\mathbf{F} \mid x_i, y_i)$ →103

What we need

- an optimization algorithm for

$$\mathbf{F}^* = \arg\min_{\mathbf{F}} \sum_{i=1}^{k} e_i^2(\mathbf{F} \mid X)$$

- the 7-point estimate is a good starting point $\mathbf{F}_0$

Levenberg-Marquardt (LM) Iterative Estimation in a Nutshell

Consider error function $\mathbf{e}_i(\boldsymbol{\theta}) = f(\mathbf{x}_i, \mathbf{y}_i, \boldsymbol{\theta}) \in \mathbb{R}^m$, with $\mathbf{x}_i, \mathbf{y}_i$ given, $\boldsymbol{\theta} \in \mathbb{R}^q$ unknown

$\theta = \mathbf{F}$, $q = \cancel{9}$ 7, $m = 1$ for f.m. estimation

Our goal: $\boldsymbol{\theta}^* = \arg\min_{\boldsymbol{\theta}} \sum_{i=1}^{k} \|\mathbf{e}_i(\boldsymbol{\theta})\|^2$

Idea 1 (Gauss-Newton approximation): proceed iteratively for $s = 0, 1, 2, \ldots$

$$\boldsymbol{\theta}^{s+1} := \boldsymbol{\theta}^s + \mathbf{d}_s\,, \quad \text{where} \quad \mathbf{d}_s = \arg\min_{\mathbf{d}} \sum_{i=1}^{k} \|\mathbf{e}_i(\boldsymbol{\theta}^s + \mathbf{d})\|^2 \tag{19}$$

$$\mathbf{e}_i(\boldsymbol{\theta}^s + \mathbf{d}) \approx \mathbf{e}_i(\boldsymbol{\theta}^s) + \mathbf{L}_i\,\mathbf{d},$$

$\| e_i(\theta^s) + L_i d \|^2$

$$(\mathbf{L}_i)_{jl} = \frac{\partial\big(\mathbf{e}_i(\boldsymbol{\theta})\big)_j}{\partial(\boldsymbol{\theta})_l}, \qquad \mathbf{L}_i \in \mathbb{R}^{m,q}$$

typically a long matrix

Then the solution to Problem (19) is a set of normal eqs

$$-\underbrace{\sum_{i=1}^{k} \mathbf{L}_i^\top \mathbf{e}_i(\boldsymbol{\theta}^s)}_{\mathbf{e}\in\mathbb{R}^{q,1}} = \underbrace{\left(\sum_{i=1}^{k} \mathbf{L}_i^\top \mathbf{L}_i\right)}_{\mathbf{L}\in\mathbb{R}^{q,q}} \mathbf{d}_s, \tag{20}$$

$d_s = -L \backslash e$

- $\mathbf{d}_s$ can be solved for by Gaussian elimination using Choleski decomposition of $\mathbf{L}$

  $\mathbf{L}$ symmetric $\Rightarrow$ use Choleski, almost $2\times$ faster than Gauss-Seidel, see bundle adjustment $\rightarrow$137
- such updates do not lead to stable convergence $\longrightarrow$ ideas of Levenberg and Marquardt

LM (cont'd)

Idea 2 (Levenberg): replace $\sum_i \mathbf{L}_i^\top \mathbf{L}_i$ with $\sum_i \mathbf{L}_i^\top \mathbf{L}_i + \lambda\,\mathbf{I}$ for some damping factor $\lambda \geq 0$

Idea 3 (Marquardt): replace $\lambda\,\mathbf{I}$ with $\lambda \sum_i \operatorname{diag}(\mathbf{L}_i^\top \mathbf{L}_i)$ to adapt to local curvature:

$$-\sum_{i=1}^{k} \mathbf{L}_i^\top \mathbf{e}_i(\boldsymbol{\theta}^s) = \left(\sum_{i=1}^{k} \bigl(\mathbf{L}_i^\top \mathbf{L}_i + \lambda \operatorname{diag}(\mathbf{L}_i^\top \mathbf{L}_i) \bigr) \right) \mathbf{d}_s$$

Idea 4 (Marquardt): adaptive λ small $\lambda \to$ Gauss-Newton, large $\lambda \to$ gradient descend

1. choose $\lambda \approx 10^{-3}$ and compute $\mathbf{d}_s$
2. if $\sum_i \|\mathbf{e}_i(\boldsymbol{\theta}^s + \mathbf{d}_s)\|^2 < \sum_i \|\mathbf{e}_i(\boldsymbol{\theta}^s)\|^2$ then accept $\mathbf{d}_s$ and set $\lambda := \lambda/10$, $s := s+1$
3. otherwise set $\lambda := 10\lambda$ and recompute $\mathbf{d}_s$

- sometimes different constants are needed for the 10 and 10^{-3}
- note that $\mathbf{L}_i \in \mathbb{R}^{m,q}$ (long matrix) but each contribution $\mathbf{L}_i^\top \mathbf{L}_i$ is a square singular $q \times q$ matrix (always singular for $k < q$)
- error can be made robust to outliers, see the trick →111
- we have approximated the least squares Hessian by ignoring second derivatives of the error function (Gauss-Newton approximation) See [Triggs et al. 1999, Sec. 4.3]
- λ helps avoid the consequences of gauge freedom →139
- modern variants of LM are Trust Region methods

LM with Sampson Error for Fundamental Matrix Estimation

Sampson (derived by linearization over point coordinates u^1, v^1, u^2, v^2)

$$e_i(\mathbf{F}) = \frac{\varepsilon_i}{\|\mathbf{J}_i\|} = \frac{\underline{\mathbf{y}}_i^\top \mathbf{F} \underline{\mathbf{x}}_i}{\sqrt{\|\mathbf{S}\mathbf{F}\underline{\mathbf{x}}_i\|^2 + \|\mathbf{S}\mathbf{F}^\top \underline{\mathbf{y}}_i\|^2}} \quad \text{where} \quad \mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

LM (by linearization over parameters $\mathbf{F}$)

$$\mathbf{L}_i = \frac{\partial e_i(\mathbf{F})}{\partial \mathbf{F}} = \cdots = \frac{1}{2\|\mathbf{J}_i\|} \left[\left(\underline{\mathbf{y}}_i - \frac{2e_i}{\|\mathbf{J}_i\|} \mathbf{S}\mathbf{F}\underline{\mathbf{x}}_i \right) \underline{\mathbf{x}}_i^\top + \underline{\mathbf{y}}_i \left(\underline{\mathbf{x}}_i - \frac{2e_i}{\|\mathbf{J}_i\|} \mathbf{S}\mathbf{F}^\top \underline{\mathbf{y}}_i \right)^\top \right] \tag{21}$$

- $\mathbf{L}_i$ in (21) is a 3×3 matrix, must be reshaped to dimension-9 vector $\mathrm{vec}(\mathbf{L}_i)$ to be used in LM
- $\underline{\mathbf{x}}_i$ and $\underline{\mathbf{y}}_i$ in Sampson error are normalized to unit homogeneous coordinate (21) relies on this
- reinforce $\operatorname{rank} \mathbf{F} = 2$ after each LM update to stay in the fundamental matrix manifold and $\|\mathbf{F}\| = 1$ to avoid gauge freedom by SVD →109
- LM linearization could be done by numerical differentiation (small dimension)

▶Local Optimization for Fundamental Matrix Estimation

Given a set $X = \{(x_i, y_i)\}_{i=1}^{k}$ of $k \gg 7$ inlier correspondences, compute a statistically efficient estimate for fundamental matrix $\mathbf{F}$.

Summary so far

1. Find the conditioned (→91) 7-point $\mathbf{F}_0$ (→83) from a suitable 7-tuple
2. Improve the $\mathbf{F}_0^*$ using the LM optimization (→106–107) and the Sampson error (→108) on all inliers, reinforce rank-2, unit-norm $\mathbf{F}_k^*$ after each LM iteration using SVD

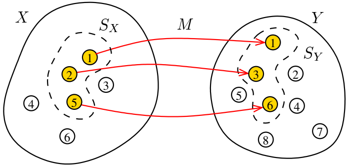
We are not yet done

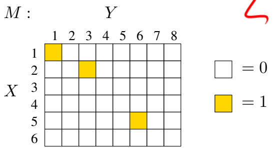
- if there are no wrong correspondences (mismatches, outliers), this gives a local optimum given the 7-point initial estimate
- the algorithm breaks under contamination of (inlier) correspondences by outliers
- the full problem involves finding the inliers!
- in addition, we need a mechanism for jumping out of local minima (and exploring the space of all fundamental matrices)

▶The Full Problem of Matching and Fundamental Matrix Estimation

Problem: Given image point sets $X = \{x_i\}_{i=1}^m$ and $Y = \{y_j\}_{j=1}^n$ and their descriptors D, find the most probable

1. inliers $S_X \subseteq X$, $S_Y \subseteq Y$
2. one-to-one perfect matching $M\colon S_X \to S_Y$ — perfect matching: 1-factor of the bipartite graph
3. fundamental matrix $\mathbf{F}$ such that $\operatorname{rank}\mathbf{F} = 2$
4. such that for each $x_i \in S_X$ and $y_j = M(x_i)$ it is probable that
   a) the image descriptor $D(x_i)$ is similar to $D(y_j)$, and
   b) the total geometric error $E = \sum_{ij} e_{ij}^2(\mathbf{F})$ is small — note a slight change in notation: e_{ij}
5. inlier-outlier and outlier-outlier matches are improbable





$2^{m \cdot n}$

$$(M^*, \mathbf{F}^*) = \arg\max_{M,\mathbf{F}} p(E, D, \mathbf{F} \mid M)\, P(M) \qquad (22)$$

- probabilistic model: an efficient language for problem formulation — it also unifies 4.a and 4.b
- the (22) is a Bayesian probabilistic model — there is a constant number of random variables!
- binary matching table $M_{ij} \in \{0, 1\}$ of fixed size $m \times n$
  - each row/column contains at most one unity
  - zero rows/columns correspond to unmatched point x_i/y_j

Deriving A Robust Matching Model by Approximate Marginalization

For algorithmic efficiency, instead of $(M^*, \mathbf{F}^*) = \arg\max_{M,\mathbf{F}} p(E, D, \mathbf{F} \mid M)\, P(M)$ solve

$p(E, D, F, M) =$

$$\boxed{\mathbf{F}^* = \arg\max_{\mathbf{F}} p(E, D, \mathbf{F})} \tag{23}$$

by marginalization of $p(E, D, \mathbf{F} \mid M)\, P(M)$ over M — this changes the problem!

ignoring that M are 1:1 matchings and assuming correspondence-wise independence:

$$p(E, D, \mathbf{F} \mid M) P(M) = \prod_{i=1}^{m} \prod_{j=1}^{n} p_e(e_{ij}, d_{ij}, \mathbf{F} \mid m_{ij}) P(m_{ij})$$

- e_{ij} represents geometric error for match $x_i \leftrightarrow y_i$: $e_{ij}(x_i, y_i, \mathbf{F})$
- d_{ij} represents descriptor similarity for match $x_i \leftrightarrow y_i$: $d_{ij} = \|\mathbf{d}(x_i) - \mathbf{d}(y_j)\|$

Marginalization: — ignore that M is a matching and take all 2^{mn} terms

$$p(E, D, \mathbf{F}) \approx \sum_{m_{11} \in \{0,1\}} \sum_{m_{12}} \cdots \sum_{m_{mn}} p(E, D, \mathbf{F} \mid M) P(M) =$$

$$= \sum_{m_{11}} \cdots \sum_{m_{mn}} \prod_{i=1}^{m} \prod_{j=1}^{n} p_e(e_{ij}, d_{ij}, \mathbf{F} \mid m_{ij}) P(m_{ij}) \overset{\circledast 1}{=} \cdots =$$

$$= \prod_{i=1}^{m} \prod_{j=1}^{n} \underbrace{\sum_{m_{ij} \in \{0,1\}} p_e(e_{ij}, d_{ij}, \mathbf{F} \mid m_{ij}) P(m_{ij})}_{\text{we will continue with this term}}$$

Robust Matching Model (cont'd)

$$\sum_{m_{ij}\in\{0,1\}} p_e(e_{ij}, d_{ij}, \mathbf{F} \mid m_{ij}) P(m_{ij}) =$$

$$= \underbrace{p_e(e_{ij}, d_{ij}, \mathbf{F} \mid m_{ij} = 1)}_{p_1(e_{ij}, d_{ij}, \mathbf{F})} \underbrace{P(m_{ij} = 1)}_{1-P_0} + \underbrace{p_e(e_{ij}, d_{ij}, \mathbf{F} \mid m_{ij} = 0)}_{p_0(e_{ij}, d_{ij}, \mathbf{F})} \underbrace{P(m_{ij} = 0)}_{P_0} =$$

$$= \boxed{(1 \cancel{- P_0})\, p_1(e_{ij}, d_{ij}, \mathbf{F}) + P_0\, p_0(e_{ij}, d_{ij}, \mathbf{F})} \tag{24}$$

1−P

- the $p_0(e_{ij}, d_{ij}, \mathbf{F})$ is a penalty for 'missing a correspondence' but it should be a p.d.f. (cannot be a constant) (→113 for a simplification)

$$\text{choose} \quad P_0 \to 1, \quad p_0(\cdot) \to 0 \quad \text{so that} \quad \frac{P_0}{1 - P_0}\, p_0(\cdot) \approx \text{const}$$

- the $p_1(e_{ij}, d_{ij}, \mathbf{F})$ is typically an easy-to-design term: assuming independence of geometric error and descriptor similarity:

$$p_1(e_{ij}, d_{ij}, \mathbf{F}) = p_1(e_{ij} \mid \mathbf{F})\, p_F(\mathbf{F})\, p_1(d_{ij})$$

- we choose, eg.

$$p_1(e_{ij} \mid \mathbf{F}) = \frac{1}{T_e(\sigma_1)}\, e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_1^2}}, \quad p_1(d_{ij}) = \frac{1}{T_d(\sigma_d, \dim \mathbf{d})}\, e^{-\frac{\|\mathbf{d}(x_i) - \mathbf{d}(y_j)\|^2}{2\sigma_d^2}} \tag{25}$$

- $\mathbf{F}$ is a random variable and σ_1, σ_d, P_0 are parameters
- the form of $T(\sigma_1)$ depends on error definition, it may depend on x_i, y_j but not on $\mathbf{F}$
- we will continue with the result from (24)

▶Simplified Robust Energy (Error) Function

- assuming the choice of p_1 as in (25), we are simplifying

$$p(E,D,\mathbf{F}) = p(E,D \mid \mathbf{F})\, p_F(\mathbf{F}) = \\ = p_F(\mathbf{F}) \prod_{i=1}^{m} \prod_{j=1}^{n} \Big[(1-P_0)\, p_1(e_{ij}, d_{ij} \mid \mathbf{F}) + P_0\, p_0(e_{ij}, d_{ij} \mid \mathbf{F})\Big]$$

- we choose $\sigma_0 \gg \sigma_1$ and omit d_{ij} for simplicity; then the square-bracket term is

$$\frac{1-P_0}{T_e(\sigma_1)}\, e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_1^2}} + \frac{P_0}{T_e(\sigma_0)}\, e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_0^2}}$$

- we define the 'potential function' as: $V(x) = -\log p(x)$, then

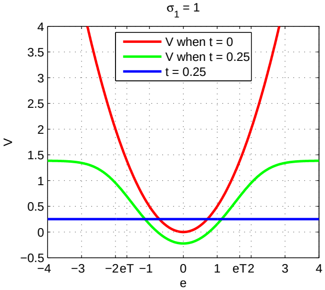
$$V(E,D \mid \mathbf{F}) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\underbrace{-\log\frac{1-P_0}{T_e(\sigma_1)}}_{\Delta\,=\,\mathrm{const}} - \log\left(e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_1^2}} + \underbrace{\frac{P_0}{1-P_0}\,\frac{T_e(\sigma_1)}{T_e(\sigma_0)}\, e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_0^2}}}_{t\,\approx\,\mathrm{const}}\right)\right] =$$

$$= \boxed{m\,n\,\Delta + \sum_{i=1}^{m}\sum_{j=1}^{n} \underbrace{-\log\left(e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_1^2}} + t\right)}_{\hat{V}(e_{ij})}} \qquad (26)$$

- note we are summing over all $m\,n$ matches (m, n are constant!)

▶The Action of the Robust Matching Model on Data

Example for $\hat{V}(e)$ from (26):



red – the usual (non-robust) error

blue – the rejected correspondence penalty t

green – 'robust energy' (26)

when $t = 0$

- if the error of a correspondence exceeds a limit, it is ignored
- then $\hat{V}(e) = \text{const}$ and we essentially count outliers in (26)
- t controls the 'turn-off' point
- the inlier/outlier threshold is e_T – the error for which $(1 - P_0)\, p_1(e_T) = P_0\, p_0(e_T)$: note that $t \approx 0$

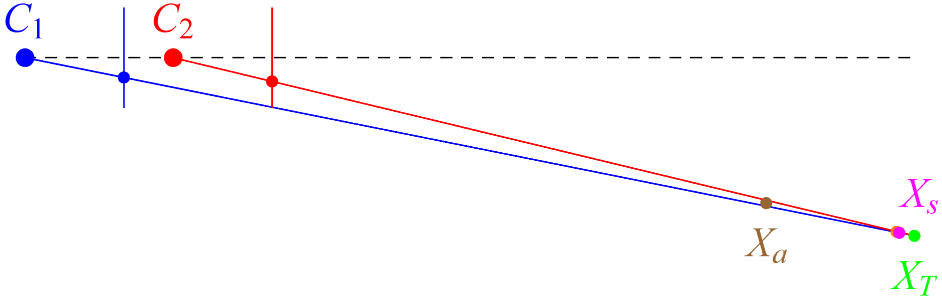
$$e_T = \sigma_1 \sqrt{-\log t^2} \tag{27}$$

The full optimization problem (23) uses (26):

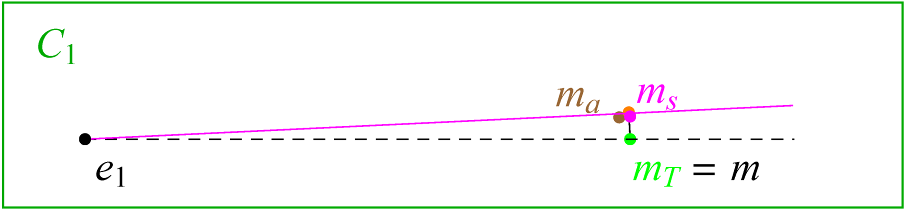
$$\mathbf{F}^* = \arg\max_{\mathbf{F}} \frac{\overbrace{p(E, D \mid \mathbf{F})}^{\text{data model}} \cdot \overbrace{p(\mathbf{F})}^{\text{prior}}}{\underbrace{p(E, D)}_{\text{evidence}}} \approx \boxed{\arg\min_{\mathbf{F}} \left[V(\mathbf{F}) + \sum_{i=1}^{m} \sum_{j=1}^{n} \log\left(e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_1^2}} + t \right) \right]}$$

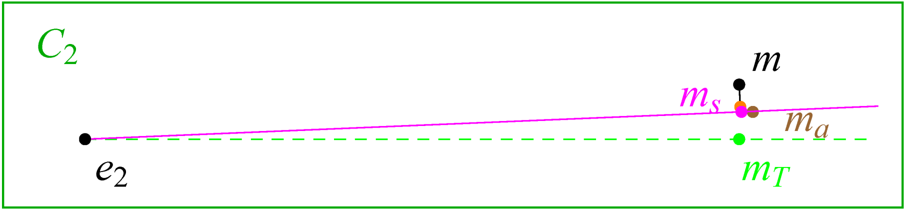
- typically we take $V(\mathbf{F}) = -\log p(\mathbf{F}) = 0$ unless we need to stabilize a computation, e.g. when video camera moves smoothly (on a high-mass vehicle) and we have a prediction for $\mathbf{F}$
- evidence is not needed unless we want to compare different models (eg. homography vs. epipolar geometry)

Thank You



C_1
C_2
X_s
X_a
X_T


C_1
m_a
m_s
e_1
$m_T = m$


C_2
m
m_s
m_a
e_2
m_T