▶The Triangulation Problem

Problem: Given cameras $\mathbf{P}_1$, $\mathbf{P}_2$ and a correspondence $x \leftrightarrow y$ compute a 3D point $\mathbf{X}$ projecting to x and y

$$\lambda_1\,\underline{\mathbf{x}} = \mathbf{P}_1\underline{\mathbf{X}}\,, \qquad \lambda_2\,\underline{\mathbf{y}} = \mathbf{P}_2\underline{\mathbf{X}}\,, \qquad \underline{\mathbf{x}} = \begin{bmatrix} u^1 \\ v^1 \\ 1 \end{bmatrix}, \qquad \underline{\mathbf{y}} = \begin{bmatrix} u^2 \\ v^2 \\ 1 \end{bmatrix}, \qquad \mathbf{P}_i = \begin{bmatrix} (\mathbf{p}_1^i)^\top \\ (\mathbf{p}_2^i)^\top \\ (\mathbf{p}_3^i)^\top \end{bmatrix}$$

Linear triangulation method

$$u^1\,(\mathbf{p}_3^1)^\top\underline{\mathbf{X}} = (\mathbf{p}_1^1)^\top\underline{\mathbf{X}}, \qquad u^2\,(\mathbf{p}_3^2)^\top\underline{\mathbf{X}} = (\mathbf{p}_1^2)^\top\underline{\mathbf{X}},$$

$$v^1\,(\mathbf{p}_3^1)^\top\underline{\mathbf{X}} = (\mathbf{p}_2^1)^\top\underline{\mathbf{X}}, \qquad v^2\,(\mathbf{p}_3^2)^\top\underline{\mathbf{X}} = (\mathbf{p}_2^2)^\top\underline{\mathbf{X}},$$

Gives

$$\mathbf{D}\underline{\mathbf{X}} = \mathbf{0}, \qquad \mathbf{D} = \begin{bmatrix} u^1\,(\mathbf{p}_3^1)^\top - (\mathbf{p}_1^1)^\top \\ v^1\,(\mathbf{p}_3^1)^\top - (\mathbf{p}_2^1)^\top \\ u^2\,(\mathbf{p}_3^2)^\top - (\mathbf{p}_1^2)^\top \\ v^2\,(\mathbf{p}_3^2)^\top - (\mathbf{p}_2^2)^\top \end{bmatrix}, \qquad \mathbf{D} \in \mathbb{R}^{4,4}, \quad \underline{\mathbf{X}} \in \mathbb{R}^4 \tag{14}$$

- back-projected rays will generally not intersect due to image error, see next
- using Jack-knife (→63) not recommended — sensitive to small error
- we will use SVD (→89)
- but the result will not be invariant to projective frame

  replacing $\mathbf{P}_1 \mapsto \mathbf{P}_1\mathbf{H}$, $\mathbf{P}_2 \mapsto \mathbf{P}_2\mathbf{H}$ does not always result in $\underline{\mathbf{X}} \mapsto \mathbf{H}^{-1}\underline{\mathbf{X}}$
- note the homogeneous form in (14) can represent points at infinity

▶The Least-Squares Triangulation by SVD

- if $\mathbf{D}$ is full-rank we may minimize the algebraic least-squares error

$$\boldsymbol{\varepsilon}^2(\underline{\mathbf{X}}) = \|\mathbf{D}\underline{\mathbf{X}}\|^2 \quad \text{s.t.} \quad \|\underline{\mathbf{X}}\| = 1, \qquad \underline{\mathbf{X}} \in \mathbb{R}^4$$

- let $\mathbf{D}_i$ be the i-th row of $\mathbf{D}$, then

$$\|\mathbf{D}\underline{\mathbf{X}}\|^2 = \sum_{i=1}^{4} (\mathbf{D}_i\, \underline{\mathbf{X}})^2 = \sum_{i=1}^{4} \underline{\mathbf{X}}^\top \mathbf{D}_i^\top \mathbf{D}_i\, \underline{\mathbf{X}} = \underline{\mathbf{X}}^\top \mathbf{Q}\, \underline{\mathbf{X}}, \text{ where } \mathbf{Q} = \sum_{i=1}^{4} \mathbf{D}_i^\top \mathbf{D}_i = \mathbf{D}^\top \mathbf{D} \ \in \mathbb{R}^{4,4}$$

- we write the SVD of $\mathbf{Q}$ as $\mathbf{Q} = \sum_{j=1}^{4} \sigma_j^2\, \mathbf{u}_j \mathbf{u}_j^\top$, in which [Golub & van Loan 2013, Sec. 2.5]

$$\sigma_1^2 \geq \cdots \geq \sigma_4^2 \geq 0 \quad \text{and} \quad \mathbf{u}_l^\top \mathbf{u}_m = \begin{cases} 0 & \text{if } l \neq m \\ 1 & \text{otherwise} \end{cases}$$

- then $\underline{\mathbf{X}} = \arg \min_{\mathbf{q}, \|\mathbf{q}\|=1} \mathbf{q}^\top \mathbf{Q}\, \mathbf{q} = \mathbf{u}_4$

Proof (by contradiction).

Let $\bar{\mathbf{q}} = \sum_{i=1}^{4} a_i \mathbf{u}_i$ s.t. $\sum_{i=1}^{4} a_i^2 = 1$, then $\|\bar{\mathbf{q}}\| = 1$, and

$$\bar{\mathbf{q}}^\top \mathbf{Q}\, \bar{\mathbf{q}} = \sum_{j=1}^{4} \sigma_j^2\, \bar{\mathbf{q}}^\top \mathbf{u}_j\, \mathbf{u}_j^\top \bar{\mathbf{q}} = \sum_{j=1}^{4} \sigma_j^2\, (\mathbf{u}_j^\top \bar{\mathbf{q}})^2 = \cdots = \sum_{j=1}^{4} a_j^2 \sigma_j^2 \ \geq \sum_{j=1}^{4} a_j^2 \sigma_4^2 = \sigma_4^2$$

□

▶cont'd

- if $\sigma_4 \ll \sigma_3$, there is a unique solution $\underline{\mathbf{X}} = \mathbf{u}_4$ with residual error $(\mathbf{D}\,\underline{\mathbf{X}})^2 = \sigma_4^2$
  the quality (conditioning) of the solution may be expressed as $q = \sigma_3/\sigma_4$ (greater is better)

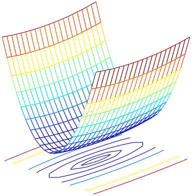
Matlab code for the least-squares solver:

```
[U,O,V] = svd(D);
X = V(:,end);
q = O(3,3)/O(4,4);
```

⊛ P1; 1pt: Why did we decompose $\mathbf{D}$ and not $\mathbf{Q} = \mathbf{D}^\top\mathbf{D}$?

▶Numerical Conditioning

- The equation $\mathbf{D}\underline{\mathbf{X}} = \mathbf{0}$ in (14) may be ill-conditioned for numerical computation, which results in a poor estimate for $\underline{\mathbf{X}}$.



Why: on a row of $\mathbf{D}$ there are big entries together with small entries, e.g. of orders projection centers in mm, image points in px

$$\begin{bmatrix} 10^3 & 0 & 10^3 & 10^6 \\ 0 & 10^3 & 10^3 & 10^6 \\ 10^3 & 0 & 10^3 & 10^6 \\ 0 & 10^3 & 10^3 & 10^6 \end{bmatrix}$$

Quick fix:

1. re-scale the problem by a regular diagonal conditioning matrix $\mathbf{S} \in \mathbb{R}^{4,4}$

$$\mathbf{0} = \mathbf{D}\,\underline{\mathbf{X}} = \mathbf{D}\,\mathbf{S}\,\mathbf{S}^{-1}\underline{\mathbf{X}} = \bar{\mathbf{D}}\,\underline{\bar{\mathbf{X}}}$$

choose $\mathbf{S}$ to make the entries in $\hat{\mathbf{D}}$ all smaller than unity in absolute value:

$$\mathbf{S} = \mathrm{diag}(10^{-3}, 10^{-3}, 10^{-3}, 10^{-6})$$ `S = diag(1./max(abs(D), 1))`

2. solve for $\underline{\bar{\mathbf{X}}}$ as before
3. get the final solution as $\underline{\mathbf{X}} = \mathbf{S}\,\underline{\bar{\mathbf{X}}}$

- when SVD is used in camera resection, conditioning is essential for success

→62

Algebraic Error vs Reprojection Error

- algebraic error (c – camera index, (u^c, v^c) – image coordinates) from SVD →90

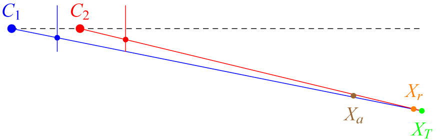
$$\varepsilon^2(\underline{\mathbf{X}}) = \sigma_4^2 = \sum_{c=1}^{2} \left[\left(u^c (\mathbf{p}_3^c)^\top \underline{\mathbf{X}} - (\mathbf{p}_1^c)^\top \underline{\mathbf{X}} \right)^2 + \left(v^c (\mathbf{p}_3^c)^\top \underline{\mathbf{X}} - (\mathbf{p}_2^c)^\top \underline{\mathbf{X}} \right)^2 \right]$$

- reprojection error

$$e^2(\underline{\mathbf{X}}) = \sum_{c=1}^{2} \left[\left(u^c - \frac{(\mathbf{p}_1^c)^\top \underline{\mathbf{X}}}{(\mathbf{p}_3^c)^\top \underline{\mathbf{X}}} \right)^2 + \left(v^c - \frac{(\mathbf{p}_2^c)^\top \underline{\mathbf{X}}}{(\mathbf{p}_3^c)^\top \underline{\mathbf{X}}} \right)^2 \right]$$

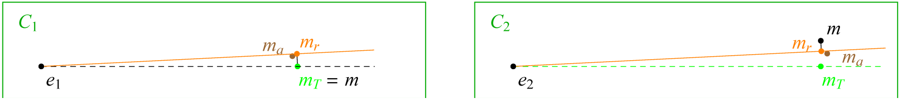
- algebraic error zero ⇔ reprojection error zero $\sigma_4 = 0 \Rightarrow$ non-trivial null space
- epipolar constraint satisfied ⇒ equivalent results
- in general: minimizing algebraic error is cheap but it gives inferior results
- minimizing reprojection error is expensive but it gives good results
- the midpoint of the common perpendicular to both optical rays gives about 50% greater error in 3D
- the golden standard method – deferred to →104

Ex:



- forward camera motion
- error $f/50$ in image 2, orthogonal to epipolar plane

X_T – noiseless ground truth position
X_r – reprojection error minimizer
X_a – algebraic error minimizer
m – measurement (m_T with noise in v^2)



►We Have Added to The ZOO

continuation from →68

problem	given	unknown	slide
camera resection	6 world–img correspondences $\left\{(X_i,\, m_i)\right\}_{i=1}^{6}$	$\mathbf{P}$	62
exterior orientation	$\mathbf{K}$, 3 world–img correspondences $\left\{(X_i,\, m_i)\right\}_{i=1}^{3}$	$\mathbf{R}$, $\mathbf{t}$	66
relative orientation	3 world-world correspondences $\left\{(X_i,\, Y_i)\right\}_{i=1}^{3}$	$\mathbf{R}$, $\mathbf{t}$	69
fundamental matrix	7 img–img correspondences $\left\{(m_i,\, m_i')\right\}_{i=1}^{7}$	$\mathbf{F}$	83
relative orientation	$\mathbf{K}$, 5 img–img correspondences $\left\{(m_i,\, m_i')\right\}_{i=1}^{5}$	$\mathbf{R}$, $\mathbf{t}$	87
triangulation	$\mathbf{P}_1$, $\mathbf{P}_2$, 1 img–img correspondence $(m_i,\, m_i')$	X	88

A bigger ZOO at `http://cmp.felk.cvut.cz/minimal/`

calibrated problems

- have fewer degenerate configurations
- can do with fewer points (good for geometry proposal generators →117)

- algebraic error optimization (SVD) makes sense in camera resection and triangulation only
- but it is not the best method; we will now focus on 'optimizing optimally'

Module V

Optimization for 3D Vision

covered by

[1] [H&Z] Secs: 11.4, 11.6, 4.7

[2] Fischler, M.A. and Bolles, R.C . Random Sample Consensus: A Paradigm for Model Fitting with Applications to Image Analysis and Automated Cartography. *Communications of the ACM* 24(6):381–395, 1981

additional references

P. D. Sampson. Fitting conic sections to 'very scattered' data: An iterative refinement of the Bookstein algorithm. *Computer Vision, Graphics, and Image Processing*, 18:97–108, 1982.

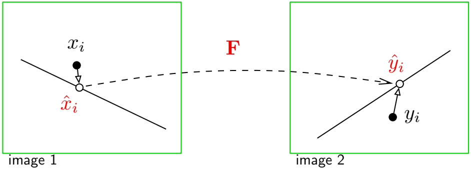
O. Chum, J. Matas, and J. Kittler. Locally optimized RANSAC. In *Proc DAGM*, LNCS 2781:236–243. Springer-Verlag, 2003.

O. Chum, T. Werner, and J. Matas. Epipolar geometry estimation via RANSAC benefits from the oriented epipolar constraint. In *Proc ICPR*, vol 1:112–115, 2004.

▶The Concept of Error for Epipolar Geometry

Problem: Given at least 8 matched points $x_i \leftrightarrow y_j$ in a general position, estimate the most likely (or most probable) fundamental matrix $\mathbf{F}$.

$$\mathbf{x}_i = (u_i^1,\, v_i^1), \quad \mathbf{y}_i = (u_i^2,\, v_i^2), \qquad i = 1, 2, \ldots, k, \quad k \geq 8$$



- detected points (measurements) $x_i,\ y_i$
- we introduce matches $\mathbf{Z}_i = (u_i^1, v_i^1, u_i^2, v_i^2) \in \mathbb{R}^4$; $\quad S = \left\{\mathbf{Z}_i\right\}_{i=1}^k$
- corrected points $\hat{x}_i,\ \hat{y}_i$; $\quad \hat{\mathbf{Z}}_i = (\hat{u}_i^1, \hat{v}_i^1, \hat{u}_i^2, \hat{v}_i^2)$; $\ \hat{S} = \left\{\hat{\mathbf{Z}}_i\right\}_{i=1}^k$ are correspondences
- correspondences satisfy the epipolar geometry exactly $\underline{\hat{\mathbf{y}}}_i^\top\, \mathbf{F}\, \underline{\hat{\mathbf{x}}}_i = 0,\ i = 1, \ldots, k$
- small correction is more probable
- let $\mathbf{e}_i(\cdot)$ be the 'reprojection error' (vector) per match i,

$$\mathbf{e}_i(x_i, y_i \mid \hat{x}_i, \hat{y}_i, \mathbf{F}) = \begin{bmatrix} \mathbf{x}_i - \hat{\mathbf{x}}_i \\ \mathbf{y}_i - \hat{\mathbf{y}}_i \end{bmatrix} = \mathbf{e}_i(\mathbf{Z}_i \mid \hat{\mathbf{Z}}_i, \mathbf{F}) = \mathbf{Z}_i - \hat{\mathbf{Z}}_i(\mathbf{F})$$

$$\|\mathbf{e}_i(\cdot)\|^2 \overset{\text{def}}{=} \mathbf{e}_i^2(\cdot) = \|\mathbf{x}_i - \hat{\mathbf{x}}_i\|^2 + \|\mathbf{y}_i - \hat{\mathbf{y}}_i\|^2 = \|\mathbf{Z}_i - \hat{\mathbf{Z}}_i(\mathbf{F})\|^2 \tag{15}$$

▶cont'd

- the total reprojection error (of all data) then is

$$L(S \mid \hat{S}, \mathbf{F}) = \sum_{i=1}^{k} \mathbf{e}_i^2(x_i, y_i \mid \hat{x}_i, \hat{y}_i, \mathbf{F}) \quad = \sum_{i=1}^{k} \mathbf{e}_i^2(\mathbf{Z}_i \mid \hat{\mathbf{Z}}_i, \mathbf{F})$$

- and the optimization problem is

$$(\hat{S}^*, \mathbf{F}^*) = \arg \min_{\substack{\mathbf{F} \\ \operatorname{rank} \mathbf{F} = 2}} \; \min_{\substack{\hat{S} \\ \underline{\hat{\mathbf{y}}}_i^\top \mathbf{F} \, \underline{\hat{\mathbf{x}}}_i = 0}} \sum_{i=1}^{k} \mathbf{e}_i^2(x_i, y_i \mid \hat{x}_i, \hat{y}_i, \mathbf{F}) \tag{16}$$

Three possible approaches

- they differ in how the correspondences $\hat{x}_i$, $\hat{y}_i$ are obtained:
  1. direct optimization of reprojection error over all variables $\hat{S}$, $\mathbf{F}$ →97
  2. Sampson optimal correction = partial correction of $\mathbf{Z}_i$ towards $\hat{\mathbf{Z}}_i$ used in an iterative minimization over $\mathbf{F}$ →98
  3. removing $\hat{x}_i$, $\hat{y}_i$ altogether = marginalization of $L(S, \hat{S} \mid \mathbf{F})$ over $\hat{S}$ followed by minimization over $\mathbf{F}$ not covered, the marginalization is difficult

Method 1: Geometric Error Optimization

- we need to encode the constraints $\hat{\underline{\mathbf{y}}}_i\, \mathbf{F}\, \hat{\underline{\mathbf{x}}}_i = 0$, $\operatorname{rank} \mathbf{F} = 2$
- idea: reconstruct 3D point via equivalent projection matrices and use reprojection error
- equivalent projection matrices are see [H&Z,Sec. 9.5] for complete characterization

$$\mathbf{P}_1 = \begin{bmatrix}\mathbf{I} & \mathbf{0}\end{bmatrix}, \quad \mathbf{P}_2 = \begin{bmatrix}[\underline{\mathbf{e}}_2]_\times \mathbf{F} + \underline{\mathbf{e}}_2 \underline{\mathbf{e}}_1^\top & \underline{\mathbf{e}}_2\end{bmatrix} \tag{17}$$

⊛ H3; 2pt: Assuming $\underline{\mathbf{e}}_1$, $\underline{\mathbf{e}}_2$ are epipoles of $\mathbf{F}$, verify that $\mathbf{F}$ is a fundamental matrix of $\mathbf{P}_1$, $\mathbf{P}_2$.
Hint: $\mathbf{A}$ is skew symmetric iff $\mathbf{x}^\top \mathbf{A} \mathbf{x} = 0$ for all vectors $\mathbf{x}$.

---

1. compute $\mathbf{F}^{(0)}$ by the 7-point algorithm →83; construct camera $\mathbf{P}_2^{(0)}$ from $\mathbf{F}^{(0)}$ using (17)
2. triangulate 3D points $\hat{\mathbf{X}}_i^{(0)}$ from matches (x_i, y_i) for all $i = 1, \ldots, k$ →88
3. starting from $\mathbf{P}_2^{(0)}$, $\hat{\mathbf{X}}^{(0)}$ minimize the reprojection error (15)

$$(\hat{\mathbf{X}}^*, \mathbf{P}_2^*) = \arg \min_{\mathbf{P}_2,\, \hat{\mathbf{X}}} \sum_{i=1}^{k} \mathbf{e}_i^2(\mathbf{Z}_i \mid \hat{\mathbf{Z}}_i(\hat{\mathbf{X}}_i, \mathbf{P}_2))$$

where

$$\hat{\mathbf{Z}}_i = (\hat{\mathbf{x}}_i, \hat{\mathbf{y}}_i) \ \text{(Cartesian)}, \quad \hat{\underline{\mathbf{x}}}_i \simeq \mathbf{P}_1 \hat{\underline{\mathbf{X}}}_i, \ \ \hat{\underline{\mathbf{y}}}_i \simeq \mathbf{P}_2 \hat{\underline{\mathbf{X}}}_i \ \text{(homogeneous)}$$

Non-linear, non-convex problem

4. compute $\mathbf{F}$ from $\mathbf{P}_1$, $\mathbf{P}_2^*$

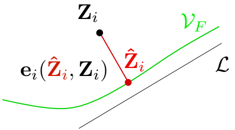
- $3k + 12$ parameters to be found: latent: $\hat{\mathbf{X}}_i$, for all i (correspondences!), non-latent: $\mathbf{P}_2$
- minimal representation: $3k + 7$ parameters, $\mathbf{P}_2 = \mathbf{P}_2(\mathbf{F})$ →143
- there are pitfalls; this is essentially bundle adjustment; we will return to this later →134

▶Method 2: First-Order Error Approximation

An elegant method for solving problems like (16):

- we will get rid of the latent parameters $\hat{X}$ needed for obtaining the correction
  [H&Z, p. 287], [Sampson 1982]
- we will recycle the algebraic error $\boldsymbol{\varepsilon} = \underline{\mathbf{y}}^\top \mathbf{F}\, \underline{\mathbf{x}}$ from →83

- consider matches $\mathbf{Z}_i$, correspondences $\hat{\mathbf{Z}}_i$, and reprojection error $\mathbf{e}_i = \|\mathbf{Z}_i - \hat{\mathbf{Z}}_i\|^2$
- correspondences satisfy $\hat{\underline{\mathbf{y}}}_i^\top \mathbf{F}\, \hat{\underline{\mathbf{x}}}_i = 0,$ $\qquad \hat{\underline{\mathbf{x}}}_i = (\hat{u}^1, \hat{v}^1, 1),\ \ \hat{\underline{\mathbf{y}}}_i = (\hat{u}^2, \hat{v}^2, 1)$
- this is a manifold $\mathcal{V}_F \in \mathbb{R}^4$: a set of points $\hat{\mathbf{Z}} = (\hat{u}^1,\ \hat{v}^1,\ \hat{u}^2,\ \hat{v}^2)$ consistent with $\mathbf{F}$
- algebraic error vanishes for $\hat{\mathbf{Z}}_i$: $\mathbf{0} = \boldsymbol{\varepsilon}_i(\hat{\mathbf{Z}}_i) = \hat{\underline{\mathbf{y}}}_i^\top \mathbf{F}\, \hat{\underline{\mathbf{x}}}_i$



Sampson's idea: Linearize the algebraic error $\boldsymbol{\varepsilon}(\mathbf{Z})$ at $\mathbf{Z}_i$ (where it is non-zero) and evaluate the resulting linear function at $\hat{\mathbf{Z}}_i$ (where it is zero). The zero-crossing replaces $\mathcal{V}_F$ by a linear manifold $\mathcal{L}$. The point on $\mathcal{V}_F$ closest to $\mathbf{Z}_i$ is replaced by the closest point on $\mathcal{L}$.

$$\boldsymbol{\varepsilon}_i(\hat{\mathbf{Z}}_i) \;\approx\; \boldsymbol{\varepsilon}_i(\mathbf{Z}_i) + \frac{\partial \boldsymbol{\varepsilon}_i(\mathbf{Z}_i)}{\partial \mathbf{Z}_i}\,(\hat{\mathbf{Z}}_i - \mathbf{Z}_i)$$

▶Sampson's Approximation of Reprojection Error

- linearize $\boldsymbol{\varepsilon}(\mathbf{Z})$ at match $\mathbf{Z}_i$, evaluate it at correspondence $\hat{\mathbf{Z}}_i$

$$0 = \boldsymbol{\varepsilon}_i(\hat{\mathbf{Z}}_i) \approx \boldsymbol{\varepsilon}_i(\mathbf{Z}_i) + \underbrace{\frac{\partial \boldsymbol{\varepsilon}_i(\mathbf{Z}_i)}{\partial \mathbf{Z}_i}}_{\mathbf{J}_i(\mathbf{Z}_i)} \underbrace{(\hat{\mathbf{Z}}_i - \mathbf{Z}_i)}_{\mathbf{e}_i(\hat{\mathbf{Z}}_i, \mathbf{Z}_i)} \overset{\text{def}}{=} \boldsymbol{\varepsilon}_i(\mathbf{Z}_i) + \mathbf{J}_i(\mathbf{Z}_i)\, \mathbf{e}_i(\hat{\mathbf{Z}}_i, \mathbf{Z}_i)$$

- goal: compute <u>function</u> $\mathbf{e}_i(\cdot)$ from $\boldsymbol{\varepsilon}_i(\cdot)$, where $\mathbf{e}_i(\cdot)$ is the distance of $\hat{\mathbf{Z}}_i$ from $\mathbf{Z}_i$
- we have a linear <u>underconstrained</u> equation for $\mathbf{e}_i(\cdot)$
- we look for a minimal $\mathbf{e}_i(\cdot)$ per match i

$$\mathbf{e}_i(\cdot)^* = \arg\min_{\mathbf{e}_i(\cdot)} \|\mathbf{e}_i(\cdot)\|^2 \quad \text{subject to} \quad \boldsymbol{\varepsilon}_i(\cdot) + \mathbf{J}_i(\cdot)\, \mathbf{e}_i(\cdot) = 0$$

- which has a closed-form solution note that $\mathbf{J}_i(\cdot)$ is not invertible! ⊛ P1; 1pt: derive $\mathbf{e}_i^*(\cdot)$

$$\begin{aligned} \mathbf{e}_i^*(\cdot) &= -\mathbf{J}_i^\top (\mathbf{J}_i \mathbf{J}_i^\top)^{-1} \boldsymbol{\varepsilon}_i(\cdot) \\ \|\mathbf{e}_i^*(\cdot)\|^2 &= \boldsymbol{\varepsilon}_i^\top(\cdot) (\mathbf{J}_i \mathbf{J}_i^\top)^{-1} \boldsymbol{\varepsilon}_i(\cdot) \end{aligned} \qquad (18)$$

- this maps $\boldsymbol{\varepsilon}_i(\cdot)$ to an estimate of $\mathbf{e}_i(\cdot)$ per correspondence
- we often do not need $\mathbf{e}_i$, just $\|\mathbf{e}_i\|^2$ exception: triangulation →104
- the unknown parameters $\mathbf{F}$ are inside: $\mathbf{e}_i = \mathbf{e}_i(\mathbf{F})$, $\boldsymbol{\varepsilon}_i = \boldsymbol{\varepsilon}_i(\mathbf{F})$, $\mathbf{J}_i = \mathbf{J}_i(\mathbf{F})$

▶Example: Fitting A Circle To Scattered Points

Problem: Fit a zero-centered circle $\mathcal{C}$ to a set of 2D points $\{x_i\}_{i=1}^k$, $\mathcal{C}\colon\ \|\mathbf{x}\|^2 - r^2 = 0$.

1. consider radial error as the 'algebraic error' $\boldsymbol{\varepsilon}(\mathbf{x}) = \|\mathbf{x}\|^2 - r^2$
2. linearize it at $\hat{\mathbf{x}}$ we are dropping i in $\boldsymbol{\varepsilon}_i$, $\mathbf{e}_i$ etc for clarity

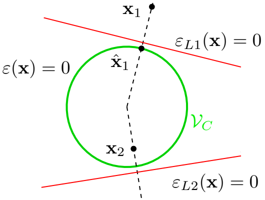
$$\boldsymbol{\varepsilon}(\hat{\mathbf{x}}) \approx \boldsymbol{\varepsilon}(\mathbf{x}) + \underbrace{\frac{\partial \boldsymbol{\varepsilon}(\mathbf{x})}{\partial \mathbf{x}}}_{\mathbf{J}(\mathbf{x})=2\mathbf{x}^\top} \underbrace{(\hat{\mathbf{x}} - \mathbf{x})}_{\mathbf{e}(\hat{\mathbf{x}},\mathbf{x})} = \cdots = 2\,\mathbf{x}^\top \hat{\mathbf{x}} - (r^2 + \|\mathbf{x}\|^2) \overset{\text{def}}{=} \boldsymbol{\varepsilon}_L(\hat{\mathbf{x}})$$

$\boldsymbol{\varepsilon}_L(\hat{\mathbf{x}}) = 0$ is a line with normal $\frac{\mathbf{x}}{\|\mathbf{x}\|}$ and intercept $\frac{r^2 + \|\mathbf{x}\|^2}{2\|\mathbf{x}\|}$ not tangent to $\mathcal{C}$, outside!

3. using (18), express error approximation $\mathbf{e}^*$ as

$$\|\mathbf{e}^*\|^2 = \boldsymbol{\varepsilon}^\top (\mathbf{J}\mathbf{J}^\top)^{-1} \boldsymbol{\varepsilon} = \frac{(\|\mathbf{x}\|^2 - r^2)^2}{4\|\mathbf{x}\|^2}$$

4. fit circle

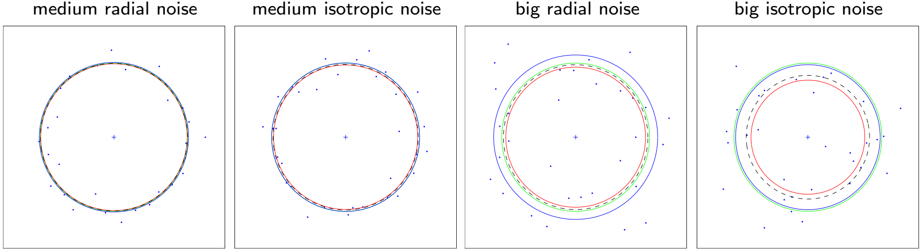


$$r^* = \arg\min_r \sum_{i=1}^k \frac{(\|\mathbf{x}_i\|^2 - r^2)^2}{4\|\mathbf{x}_i\|^2} = \cdots = \left(\frac{1}{k} \sum_{i=1}^k \frac{1}{\|\mathbf{x}_i\|^2} \right)^{-\frac{1}{2}}$$

- this example results in a convex quadratic optimization problem
- note that

$$\arg\min_r \sum_{i=1}^k (\|\mathbf{x}_i\|^2 - r^2)^2 = \left(\frac{1}{k} \sum_{i=1}^k \|\mathbf{x}_i\|^2 \right)^{\frac{1}{2}}$$

Circle Fitting: Some Results



opt: 1.8, Smp: 1.9, dir: 2.3 — 1.8, 2.0, 2.2 — 1.6, 1.8, 2.6 — 1.6, 2.0, 2.4

mean ranks over 10 000 random trials with $k = 32$ samples

green – ground truth
red – Sampson error minimizer
blue – direct radial error minimizer
black – optimal estimator for isotropic error

optimal estimator for isotropic error (black, dashed):

$$r \approx \frac{3}{4k}\sum_{i=1}^{k}\|\mathbf{x}_i\| + \sqrt{\left(\frac{3}{4k}\sum_{i=1}^{k}\|\mathbf{x}_i\|\right)^2 - \frac{1}{2k}\sum_{i=1}^{k}\|\mathbf{x}_i\|^2}$$

which method is better?

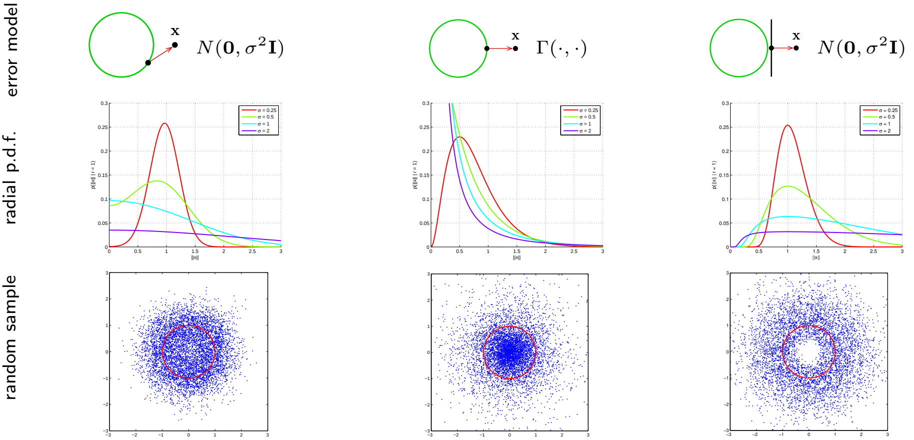
- error should model noise, radial noise and isotropic noise behave differently
- ground truth: Normally distributed isotropic error, Gamma-distributed radial error
- Sampson: better for the radial distribution model; Direct: better for the isotropic model
- no matter how corrected, the algebraic error minimizer is not an unbiased parameter estimator

Cramér-Rao bound tells us how close one can get with unbiased estimator and given k

Discussion: On The Art of Probabilistic Model Design. . .

- a few models for fitting zero-centered circle C of radius r to points in $\mathbb{R}^2$

marginalized over C | orthogonal deviation from C | Sampson approximation



$p(\mathbf{x} \mid r)$

marginalized over C

$$\approx \frac{1}{\sigma\sqrt{(2\pi)^3 r\,\|\mathbf{x}\|}}\, e^{-\frac{(\|\mathbf{x}\|-r)^2}{2\sigma^2}}$$

- mode inside the circle
- models the inside well
- tends to normal distrib.

orthogonal deviation from C

$$\frac{1}{2\pi\Gamma(\frac{r^2}{\sigma})}\,\frac{1}{\|\mathbf{x}\|^2}\left(\frac{r\|\mathbf{x}\|}{\sigma}\right)^{\frac{r^2}{\sigma}} e^{-\frac{r\|\mathbf{x}\|}{\sigma}}$$

- peak at the center
- unusable for small radii
- tends to Dirac distrib.

Sampson approximation

$$\frac{1}{r\sigma\sqrt{(2\pi)^3}}\, e^{-\frac{e^2(\mathbf{x};r)}{2\sigma^2}}$$

- mode at the circle
- hole at the center
- tends to normal distrib.

▶Sampson Error for Fundamental Matrix Manifold

The epipolar algebraic error is

$$\varepsilon_i(\mathbf{F}) = \underline{\mathbf{y}}_i^\top \mathbf{F}\, \underline{\mathbf{x}}_i, \quad \mathbf{x}_i = (u_i^1,\, v_i^1), \quad \mathbf{y}_i = (u_i^2,\, v_i^2), \qquad \varepsilon_i \in \mathbb{R}$$

Let $\mathbf{F} = \begin{bmatrix} \mathbf{F}_1 & \mathbf{F}_2 & \mathbf{F}_3 \end{bmatrix}$ (per columns) $= \begin{bmatrix} (\mathbf{F}^1)^\top \\ (\mathbf{F}^2)^\top \\ (\mathbf{F}^3)^\top \end{bmatrix}$ (per rows), $\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then

Sampson

$$\mathbf{J}_i(\mathbf{F}) = \left[\frac{\partial \varepsilon_i(\mathbf{F})}{\partial u_i^1}, \frac{\partial \varepsilon_i(\mathbf{F})}{\partial v_i^1}, \frac{\partial \varepsilon_i(\mathbf{F})}{\partial u_i^2}, \frac{\partial \varepsilon_i(\mathbf{F})}{\partial v_i^2} \right] \qquad \mathbf{J}_i \in \mathbb{R}^{1,4} \qquad \text{derivatives over point coords.}$$

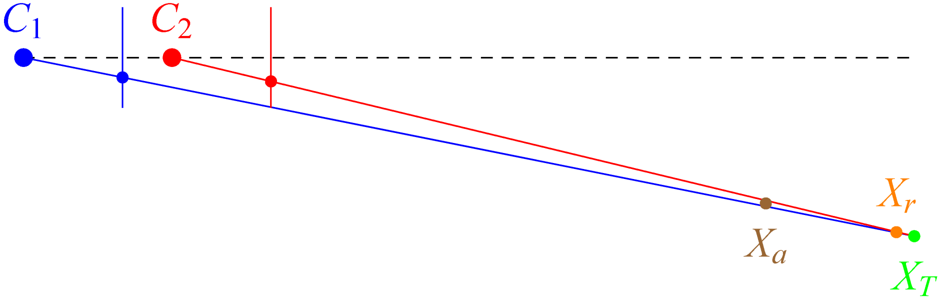
$$= \left[(\mathbf{F}_1)^\top \underline{\mathbf{y}}_i,\ (\mathbf{F}_2)^\top \underline{\mathbf{y}}_i,\ (\mathbf{F}^1)^\top \underline{\mathbf{x}}_i,\ (\mathbf{F}^2)^\top \underline{\mathbf{x}}_i \right]$$

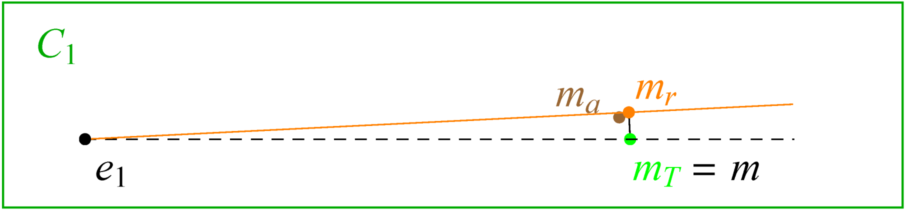
$$\mathbf{e}_i(\mathbf{F}) = -\frac{\mathbf{J}_i(\mathbf{F})\, \varepsilon_i(\mathbf{F})}{\|\mathbf{J}_i(\mathbf{F})\|^2} \qquad \mathbf{e}_i(\mathbf{F}) \in \mathbb{R}^4 \qquad \text{Sampson error vector}$$

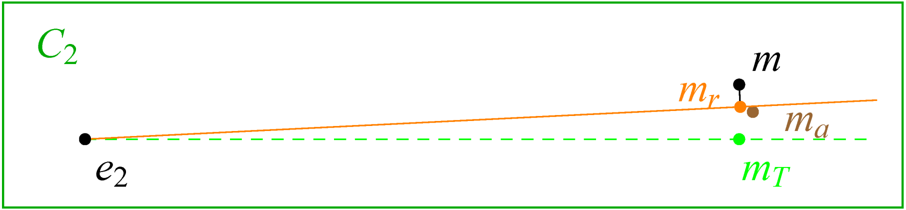
$$e_i(\mathbf{F}) = \|\mathbf{e}_i(\mathbf{F})\| = \frac{\varepsilon_i(\mathbf{F})}{\|\mathbf{J}_i(\mathbf{F})\|} = \frac{\underline{\mathbf{y}}_i^\top \mathbf{F} \underline{\mathbf{x}}_i}{\sqrt{\|\mathbf{S}\mathbf{F}\underline{\mathbf{x}}_i\|^2 + \|\mathbf{S}\mathbf{F}^\top \underline{\mathbf{y}}_i\|^2}} \qquad e_i(\mathbf{F}) \in \mathbb{R} \qquad \text{scalar Sampson error}$$

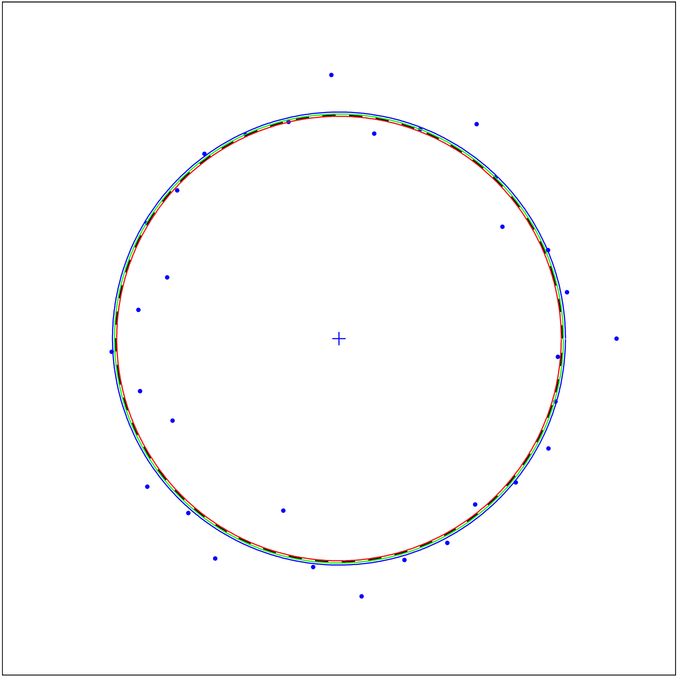
- Sampson error ‘normalizes’ the algebraic error
- automatically copes with multiplicative factors $\mathbf{F} \mapsto \lambda\mathbf{F}$
- actual optimization not yet covered →108

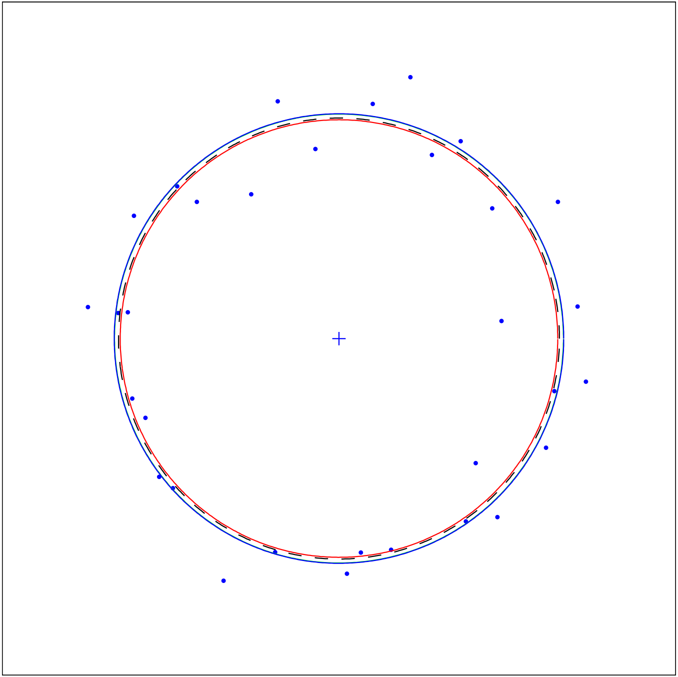
Thank You

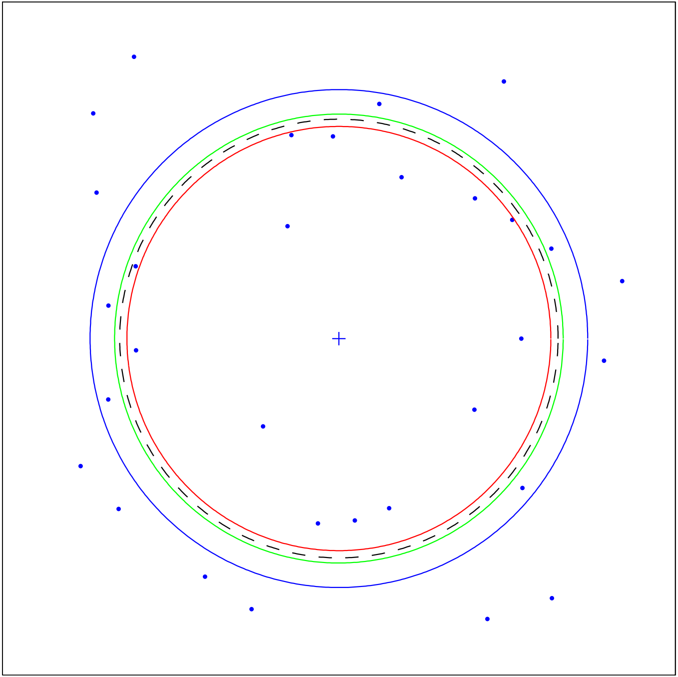

C_1
C_2
X_r
X_a
X_T

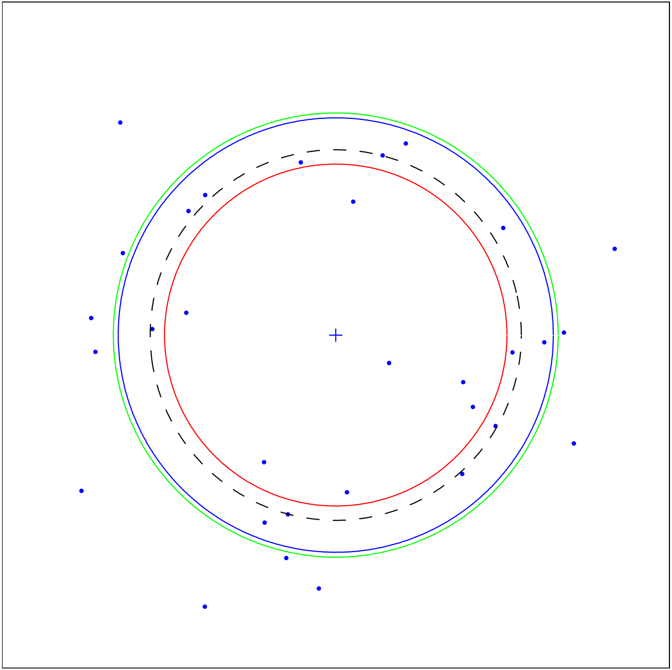

C_1
m_a
m_r
e_1
$m_T = m$

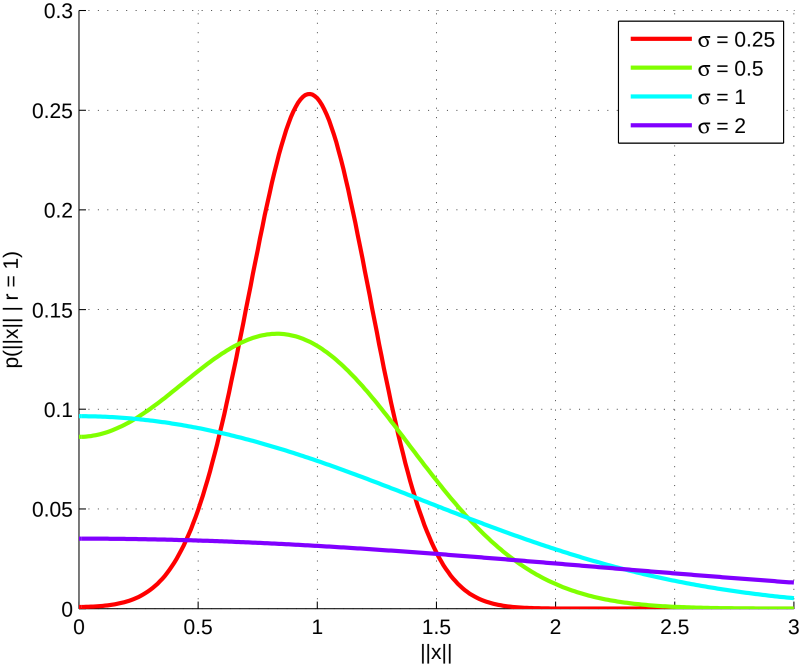

C_2
m
m_r
m_a
e_2
m_T







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σ = 0.5
σ = 1
σ = 2
p(||x|| | r = 1)
||x||
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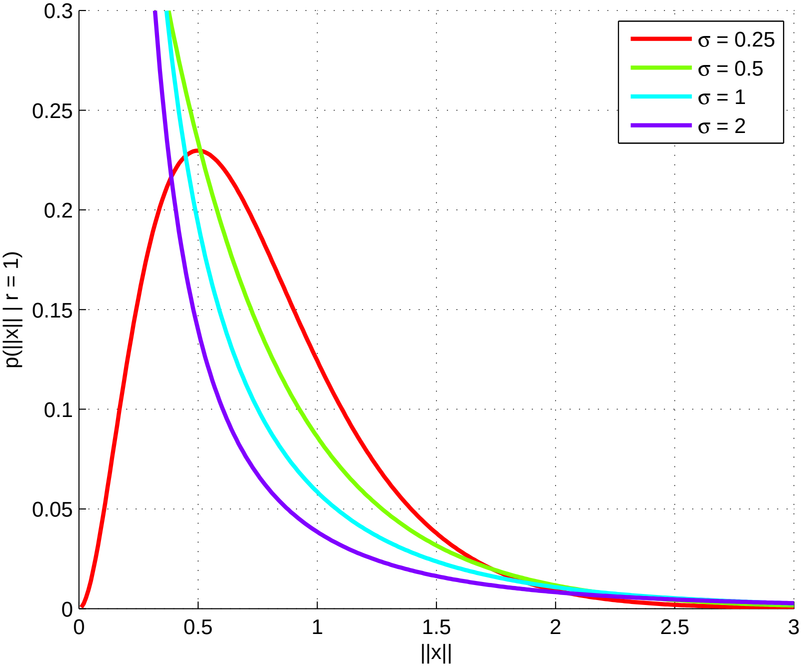
0.05
0.1
0.15
0.2
0.25
0.3
0
0.5
1
1.5
2
2.5
3



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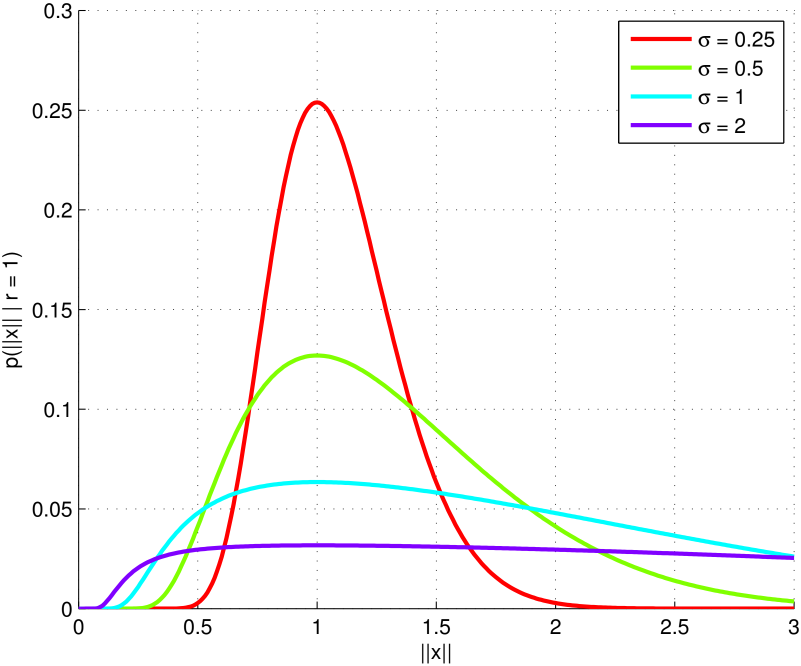

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